Examples related to Different Economic Problems:

(1) A consumer consumes two commodities Q_1 and Q_2 and the utility function is given by

$$U = 3Q_1^2 - 5Q_1Q_2 + 2Q_2^3$$

Find out marginal utilities of Q_1 and Q_2

Solution: It is given that -

$$U = 3Q_1^2 - 5Q_1Q_2 + 2Q_2^3 = f(Q_1, Q_2)$$

Since, $U = f(Q_1, Q_2)$, the marginal utilities of Q_1 and Q_2 can be derived by the use of partial derivatives of U with respect to Q_1 and Q_2 such as

Marginal Utility of $Q_1 = \frac{\partial U}{\partial Q_1}$ (assuming Q_2 as constant)

$$= \frac{\partial}{\partial Q_1}(3Q_1^2 - 5Q_1Q_2 + 2Q_2^3)$$

$$= 6Q_1 - 5Q_2 \cdot 1 + 0 \quad (\text{since } Q_2 \text{ is constant})$$

$$= 6Q_1 - 5Q_2$$

and Marginal Utility of $Q_2 = \frac{\partial U}{\partial Q_2}$ (assuming Q_1 as constant)

$$= \frac{\partial}{\partial Q_2}(3Q_1^2 - 5Q_1Q_2 + 2Q_2^3)$$

$$= 0 - 5Q_1 \cdot 1 + 6Q_2^2 \quad (\text{since } Q_1 \text{ is constant})$$

$$= 6Q_2^2 - 5Q_1$$

(2) Given the demand function as:-

$$Q_1 = 16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y$$

Find the own price elasticity, the cross price elasticity and income elasticity of the given demand functions at point $(P_1, P_2, Y) = (2, 4, 120)$. Also identify whether the goods 1 and 2 are complements or substitutes.

Solution:

It is given that -

$$Q_1 = 16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y = f(P_1, P_2, Y)$$

Since $Q_1 = f(P_1, P_2, Y)$, the own price elasticity of demand, cross elasticity of demand and income elasticity of demand can be measured by using the partial derivatives of Q_1 with respect to P_1, P_2 and Y respectively such as-

Own Price Elasticity of Demand $(E_1) = (-)\frac{\partial Q_1}{\partial P_1} \times \frac{P_1}{Q_1}$ (assuming P_2 and Y as constant)

$$\Rightarrow E_1 = (-)\frac{\partial}{\partial P_1}\left(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y\right) \times \frac{P_1}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)}$$

$$\Rightarrow E_1 = (-)(-5) \times \frac{P_1}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)}$$ (since P_2 and Y are constant)

$$\Rightarrow E_1 = \frac{5P_1}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)}$$

Now, E_1 of the given demand function at Point $(P_1, P_2, Y) = (2, 4, 120)$ -

$$E_1 = \frac{5 \cdot 2}{(16 - 5 \cdot 2 - \frac{1}{2} \cdot 4 + \frac{1}{3} \cdot 120)}$$

$$= \frac{10}{(16 - 10 - 2 + 40)} = \frac{10}{44} = 0{\cdot}23$$

Since $E_1 < 1$, hence the commodity is necessity in nature.

Cross Price Elasticity of Demand $(E_2) = \frac{\partial Q_1}{\partial P_2} \times \frac{P_2}{Q_1}$ (assuming P_1 and Y as constant)

$$\Rightarrow E_2 = \frac{\partial}{\partial P_2}\left(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y\right) \times \frac{P_2}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)}$$

$$\Rightarrow E_2 = \left(-\frac{1}{2}\right) \times \frac{P_2}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)}$$ (since P_1 and Y as constant)

At point $(2, 4, 120)$,

$$E_2 = (-)\frac{4}{2(16 - 5 \cdot 2 - \frac{1}{2} \cdot 4 + \frac{1}{3} \cdot 120)}$$

$$= (-)\frac{2}{44}$$

$$= -0{\cdot}045$$

Since E_2 is negative ($E_2 < 0$), hence the given commodities (1 and 2) are complementary goods.

Income Elasticity of Demand $(E_Y) = \frac{\partial Q_1}{\partial Y} \times \frac{Y}{Q_1}$ (assuming P_1 and P_2 as constant)

$$\Rightarrow E_Y = \frac{\partial}{\partial Y}\left(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y\right) \times \frac{Y}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)}$$

$$= \left(\frac{1}{3}\right) \times \frac{Y}{(16 - 5P_1 - \frac{1}{2}P_2 + \frac{1}{3}Y)} \quad \text{(since } P_1 \text{ and } P_2 \text{ are constant)}$$

At point (2, 4, 120), $E_Y = \frac{120}{3(16 - 5.2 \ \ \frac{1}{2}.4 + \frac{1}{3}.120)}$

$$= \frac{40}{44} = 0.91$$

Since E_Y is positive and less than one ($0 < E_Y < 1$), hence the commodity is a necessity.

(3) Given the demand function for Q_1 commodity is

$Q_1 = Y - 3P_1 + 2P_2$

Find, own price elasticity of demand, cross elasticity of demand and income elasticity of demand when given income $Y = 20$, $P_1 = 2$ and $P_2 = 4$.

<u>Solution:</u> It is given that

$$Q_1 = Y - 3P_1 + 2P_2 \ ; \ \text{i.e. } Q_1 = f(Y, P_1, P_2)$$

Since $Q_1 = f(Y, P_1, P_2)$, the own price elasticity of demand (E_1), cross Elasticity of Demand (E_2) and income elasticity of demand (E_Y) can be measured by using the partial derivatives of Q_1 with respect to P_1, P_2 and Y respectively such as –

Own Price Elasticity of Demand $(E_1) = -\frac{\partial Q_1}{\partial P_1} \times \frac{P_1}{Q_1}$ (assuming P_2 and Y as constant)

$$\Rightarrow E_1 = -\frac{\partial}{\partial P_1}(Y - 3P_1 + 2P_2) \times \frac{P_1}{(Y - 4P_1 + 2P_2)}$$

$$\Rightarrow E_1 = -(-3) \times \frac{P_1}{(Y - 3P_1 + 2P_2)} \quad \text{(since } P_2 \text{ and } Y \text{ are constant)}$$

$$\Rightarrow E_1 = \frac{3P_1}{Y - 3P_1 + 2P_2}$$

when, $Y = 20$, $P_1 = 2$ and $P_2 = 4$, then

$$E_1 = \frac{3.2}{20 - 3.2 + 2.4} = \frac{6}{20 - 6 + 8} = \frac{6}{22} = 0{\cdot}27$$

Since E_1 is less than one ($E_1 < 1$), hence the commodity (Q_1) is necessity in nature.

Cross Elasticity of Demand $(E_2) = \frac{\partial Q_1}{\partial P_2} \times \frac{P_2}{Q_1}$ (assuming P_1 and Y as constant)

$$\Rightarrow E_2 = \frac{\partial}{\partial P_2}(Y - 3P_1 + 2P_2) \times \frac{P_2}{(Y - 4P_1 + 2P_2)}$$

$$\Rightarrow E_2 = (2) \times \frac{P_2}{(Y - 3P_1 + 2P_2)}$$

$$\Rightarrow E_2 = \frac{2P_2}{(Y - 3P_1 + 2P_2)}$$

when $Y = 20$, $P_1 = 2$ and $P_2 = 4$, then -

$$E_2 = \frac{2 \times 4}{(20 - 3.2 + 2.4)} = \frac{8}{22} = 0{\cdot}36$$

Sine, E_2 is positive ($E_2 > 0$), hence the given commodities (1 and 2) are substitute goods.

Income Elasticity of Demand $(E_Y) = \frac{\partial Q_1}{\partial Y} \times \frac{Y}{Q_1}$ (assuming P_1 and P_2 as constant)

$$\Rightarrow E_Y = \frac{\partial}{\partial Y}(Y - 3P_1 + 2P_2) \times \frac{Y}{(Y - 3P_1 + 2P_2)}$$

$$\Rightarrow E_Y = (1) \times \frac{Y}{(Y - 3P_1 + 2P_2)}$$

$$\Rightarrow E_Y = \frac{Y}{(Y - 3P_1 + 2P_2)}$$

when, $Y = 20$, $P_1 = 2$ and $P_2 = 4$, then -

$$E_Y = \frac{20}{(20 - 3.2 + 2.4)} = \frac{20}{22} = 0{\cdot}9$$

Since E_Y is positive and less than one ($0 < E_Y < 1$), hence the commodity is a necessity in nature.

(4) Given the production function as -

$$Q = 7L^5K^{-4}$$

Verify whether the Euler's theorem is satisfied or not for the production function.

Solution: It is given that

$$Q = 7L^5K^{-4}$$

The above given production function is linear homogeneous one because the degree of homogeneity is equal to one (i.e., 5-4 = 1). Therefore, the above production function will satisfy the Euler's theorem.

It is known that Euler's theorem can be expressed with the help of an equation such as if $Q = f(L, K)$ -

$$Q = \frac{\partial Q}{\partial L} \times L + \frac{\partial Q}{\partial K} \times K \quad \text{——— (1)}$$

$\therefore$ Marginal Product of Labour $(MP_L) = \frac{\partial Q}{\partial L}$ (assuming K as constant)

$$= \frac{\partial}{\partial L}(7L^5K^{-4})$$

$$= 7K^{-4} \cdot \frac{\partial}{\partial L}(L^5)$$

$$= 7K^{-4} \cdot 5 \cdot L^{5-1} \text{ (since K is constant)}$$

$$= 7K^{-4} \cdot 5 \cdot L^4$$

$$= 35L^4K^{-4} \quad \text{——— (2)}$$

Marginal Product of Capital $(MP_K) = \frac{\partial Q}{\partial K}$ (assuming L as constant)

$$= \frac{\partial}{\partial K}(7L^5K^{-4})$$

$$= 7L^5 \frac{\partial}{\partial K}(K^{-4})$$

$$= 7 \cdot L^5 \cdot (-4) K^{-4-1}$$

$$= 7L^5(-4)K^{-5}$$

$$= (-28)L^5K^{-5} \quad \text{——— (3)}$$

Now putting the values of $\frac{\partial q}{\partial L}$ and $\frac{\partial q}{\partial K}$ in the right hand side of the equation (1), we have

$$R.HS \text{ of } eq^n(1) \Rightarrow \frac{\partial q}{\partial L} \times L + \frac{\partial q}{\partial K} \times K$$

$$= \{(35 L^4 K^{-4}).L\} + [\{(-28) L^5 K^{-5}\}.K]$$

$$= 35 L^5 K^{-4} - 28 L^5 K^{-4}$$

$$= 7 L^5 K^{-4}$$

$$= q = L.H.S.$$

Hence proved that the above given function satisfies the Euler's theorem.

(5) Verify whether the Euler's theorem is satisfied or not for the following production function (GU, 2004)

$$q = 10 K^{\frac{3}{4}} \sqrt{L}$$

Solution:

It is given that

$$q = 10 K^{\frac{3}{4}} \sqrt{L} = 10 K^{\frac{3}{4}} L^{\frac{1}{2}}$$

The above given production function is not linear homogeneous one because the degree of homogeneity is not equal to one (i.e., $\frac{3}{4} + \frac{1}{2} = \frac{5}{4} \neq 1$). Thus, the given production function cannot satisfy the Euler's theorem.

If $q = f(L, K)$, the Euler's theorem can be expressed in an equation such as –

$$q = \frac{\partial q}{\partial L} \times L + \frac{\partial q}{\partial K} \times K \quad \text{———} \quad (1)$$

$\therefore$ Marginal Product of labour $(MP_L) = \frac{\partial q}{\partial L}$ (assuming K as constant)

$$= \frac{\partial}{\partial L}\left(10 K^{\frac{3}{4}} L^{\frac{1}{2}}\right)$$

$$= 10 K^{\frac{3}{4}} . \frac{\partial}{\partial L}\left(L^{\frac{1}{2}}\right)$$

$$= 10 K^{\frac{3}{4}} . \left(\frac{1}{2}\right) L^{-\frac{1}{2}} \quad \text{(since K is constant)}$$

$$= 5 K^{\frac{3}{4}} L^{-\frac{1}{2}} \quad \text{———} \quad (2)$$

and, Marginal Product of capital $(MP_K) = \frac{\partial Q}{\partial K}$ (assuming L as constant)

$$= \frac{\partial}{\partial K}\left(10K^{\frac{3}{4}}L^{\frac{1}{2}}\right)$$

$$= 10L^{\frac{1}{2}}\frac{\partial}{\partial K}\left(K^{\frac{3}{4}}\right)$$

$$= 10L^{\frac{1}{2}}.\left(\frac{3}{4}\right)K^{\frac{3}{4}-1} \text{ (since L is constant)}$$

$$= \frac{15}{2}L^{\frac{1}{2}}.K^{-\frac{1}{4}} \quad \text{——} \quad (3)$$

Now substituting the values of $\frac{\partial Q}{\partial L}$ and $\frac{\partial Q}{\partial K}$ in the R.H.S of the equation (1), we have -

$$\text{RHS of eq}^n (1) \Rightarrow \frac{\partial Q}{\partial L}\times L + \frac{\partial Q}{\partial K}\times K$$

$$= \left\{\left(5K^{\frac{3}{4}}L^{-\frac{1}{2}}\right).L\right\} + \left\{\left(\frac{15}{2}L^{\frac{1}{2}}.K^{-\frac{1}{4}}\right).K\right\}$$

$$= 5K^{\frac{3}{4}}L^{\frac{1}{2}} + \frac{15}{2}K^{\frac{3}{4}}L^{\frac{1}{2}}$$

$$= 5K^{\frac{3}{4}}\sqrt{L} + 7.5K^{\frac{3}{4}}\sqrt{L}$$

$$= 12.5K^{\frac{3}{4}}\sqrt{L} \neq Q$$

Hence, the above given production function cannot satisfy the Euler's theorem.

(6) Verify whether Euler's theorem is satisfied or not for the following production function

$$Q = 5.\left(\sqrt[3]{L^5}.\sqrt[3]{K^{-2}}\right)$$

Solution: It is given that

$$Q = 5.\left(\sqrt[3]{L^5}.\sqrt[3]{K^{-2}}\right) = 5.L^{\frac{5}{3}}.K^{-\frac{2}{3}}$$

The above given production function is linear homogeneous one because the degree of homogeneity is equal to one (i.e., $\frac{5}{3} - \frac{2}{3} = 1$). Therefore, the given production function will satisfy the Euler's theorem.

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If $Q = f(L, K)$, the Euler's theorem can be expressed in an equation as –

$$Q = \frac{\partial Q}{\partial L} \times L + \frac{\partial Q}{\partial K} \times K \quad \text{—— (1)}$$

$\therefore$ Marginal Product of Labour $(MP_L) = \frac{\partial Q}{\partial L}$ (assuming K as constant)

$$= \frac{\partial}{\partial L}\left(5 . L^{\frac{5}{3}} K^{-\frac{2}{3}}\right)$$

$$= 5 . K^{-\frac{2}{3}} \frac{\partial}{\partial L}\left(L^{\frac{5}{3}}\right)$$

$$= 5 . K^{-\frac{2}{3}} \left(\frac{5}{3}\right) L^{\frac{5}{3}-1} \quad \text{(since K is constant)}$$

$$= \frac{25}{3} L^{\frac{2}{3}} K^{-\frac{2}{3}} \quad \text{—— (2)}$$

and, Marginal Product of Capital $(MP_K) = \frac{\partial Q}{\partial K}$ (assuming L as constant)

$$= \frac{\partial}{\partial K}\left(5 L^{\frac{5}{3}} K^{-\frac{2}{3}}\right)$$

$$= 5 L^{\frac{5}{3}} . \frac{\partial}{\partial K}\left(K^{-\frac{2}{3}}\right)$$

$$= 5 L^{\frac{5}{3}} . \left(-\frac{2}{3}\right) K^{-\frac{2}{3}-1} \quad \text{(since L is constant)}$$

$$= -\frac{10}{3} L^{\frac{5}{3}} K^{-\frac{5}{3}} \quad \text{—— (3)}$$

Now, substituting the values of $\frac{\partial Q}{\partial L}$ and $\frac{\partial Q}{\partial K}$ in the R.H.S. of the equation (1), we have –

R.H.S. of eq^n (1) $\Rightarrow \frac{\partial Q}{\partial L} \times L + \frac{\partial Q}{\partial K} \times K$

$$= \left(\frac{25}{3} L^{\frac{2}{3}} K^{-\frac{2}{3}}\right) \times L + \left(-\frac{10}{3} L^{\frac{5}{3}} K^{-\frac{5}{3}}\right) \times K$$

$$= \frac{25}{3} L^{\frac{2}{3}+1} K^{-\frac{2}{3}} - \frac{10}{3} L^{\frac{5}{3}} K^{-\frac{5}{3}+1}$$

$$= \frac{25}{3} L^{\frac{5}{3}} K^{-\frac{2}{3}} - \frac{10}{3} L^{\frac{5}{3}} K^{-\frac{2}{3}}$$

$$= \left(\frac{25-10}{3}\right) L^{\frac{5}{3}} K^{-\frac{2}{3}}$$

$$= 5 L^{\frac{5}{3}} K^{-\frac{2}{3}} = Q = LHS$$

Hence the given production function satisfies the Euler's theorem.

** Law of Variable Proportions: The production is the act or process of transformation of inputs into output, a good or service which has value and contributes to the utility of individuals. Thus, production of a commodity or service requires the simultaneous use of four factors of production – land, labour, capital and entrepreneur. The amount of production of any commodity, therefore, uniquely depends on the amounts of these four factors of production used under certain technical conditions. Thus, this technical relationship between quantities of physical inputs and quantities of output of goods can be expressed with the help of the concept production function. In other words, production function expresses the technical relationship, which relates the maximum amount of output that can be obtained from the use of a given amount of inputs.

Let us assume that Q amount of a commodity (A) is produced by using l, k, l_1 and e amounts of inputs-labour (L), capital ( ), land (L_1) and entrepreneur (E) respectively. Then, the production function can be expressed as–

$$Q = f(l, k, l_1, e) \quad \text{———} \quad (\text{I})$$

Total product (TP) is the amount of total output produced by a given amount of a factor of production, keeping the quantity of other factors fixed or constant. Assuming only labour (L) factor is varied and all other factors (K, L_1 and E) are kept constant, Q amount of output is produced by the use of 'l' amount of labour (L) factor and it is known as total product (TP) of labour (L).

Average product (AP) is the quantity of total output produced per unit of a variable factor or input, holding all other inputs fixed. It, usually abbreviated as AP, is measured by dividing total product (TP) by the quantity of the variable factor. Assuming labour (L) as variable factor, keeping all other factors (K, L_1 and E) as fixed, average product (AP) of labour (AP_L) can be derived by the ratio of total product (TP), i.e, Q, and 'l' quantity of labour (L). Thus, AP_L can be defined as $AP_L = \frac{TP}{No\text{-}of\ L} = \frac{Q}{l}$ —— (2)

Marginal product (MP) is the additional output produced as a result of employing an additional or extra unit of the variable factor or input, keeping all other factors as constant. It (MP) is the ratio of change in total product (ΔTP) and change in the variable factor (say Δl). Keeping all other factors constant, marginal product of labour (MP_L) can be defined as the ratio of change in total product (i.e., ΔQ) and change in the employment of labour (Δl). Thus, MP_L can be expressed as –

$$MP_L = \frac{\Delta TP}{\Delta\ \text{Variable Factor}} = \frac{\Delta Q}{\Delta l} \quad \text{—— (3)}$$

If these changes are very small then

$MP_L = \frac{\partial Q}{\partial l}$, i.e., the partial derivative of Q with respect to l, keeping k, l_1 and e as constant.

Keeping other factors fixed, as the amount of the variable factor increases, the total output or product also increases initially. But, the rate of increase in total product varies at different levels of employment of the factor and after a certion level of employment of the variable factor the total product starts declining. Thus, the law of variable proportions explains the behaviour of output as the quantity of the variable factor is increased, keeping the quantity of other factors fixed.

The behaviour of output when the varing quantity of one factor is combined with a fixed quantity of the other can be divided into three distinct stages, i.e, known as three stages of production – Increasing Returns, Diminishing Returns and Negative Returns. The behaviour of output in these three stages of production can be explaind by the below figure and table as –

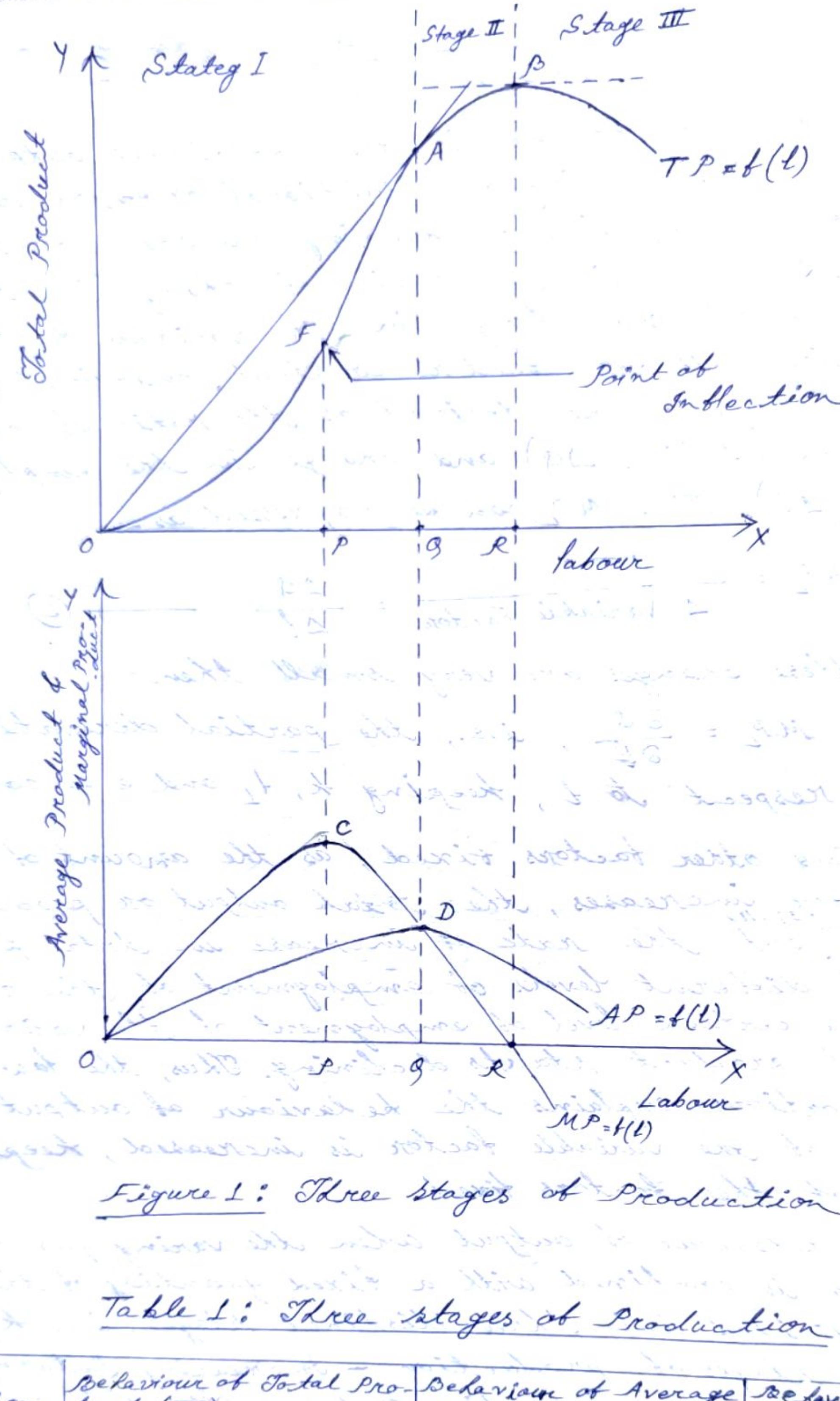

Figure 1: Three Stages of Production

Table 1: Three stages of Production

Stages of Production	Behaviour of Total Product (TP)	Behaviour of Average Product (AP)	Behaviour of Marginal Product.
Stage I: Stage of Increasing Returns.	TP increases at an increasing rate initially and after certain level of employment of the variable factor (i.e., l) TP increases at a diminishing rate.	AP increases with the increase in the employment of the variable factor (i.e., l) and reaches it's maximum point, where slope of AP is zero.	MP increases with the increase in the variable factor (i.e., l) at the initial stage and after reaching a certain level of variable factor MP starts to decline. and

	From this same level of employment of the variable factor l (i.e, OP), MP starts to decline.	Since $AP = f(l)$, hence maximization of AP requires - (a) 1st order condition - $\frac{\partial}{\partial l}(AP_L) = 0$ (b) 2nd order condition $\frac{\partial^2}{\partial l^2}(AP_L) < 0$	MP cuts the AP curve at its maximum point. At the maximum point of AP, $AP = MP$
Stage II: Stage of Diminishing Returns	The TP continues to increase at a diminishing rate until it reaches its maximum point (i.e., B) where the 2nd stage ends. Since $TP = f(l)$, hence maximization of TP requires — (a) 1st Order condition: $\frac{\partial}{\partial l}(TP_L) = 0$ (b) 2nd Order condition: $\frac{\partial^2}{\partial l^2}(TP_L) < 0$	AP starts to decline and the rate of decline is lower than MP and AP is positive. Thus, slope of AP is negative.	The MP continues to decline, but positive and the rate of decline is higher than AP. At the end of 2nd stage (at point R), MP equals to zero, corresponding to the highest level of TP (at point B).
Stage III: Stage of Negative Returns	The TP starts to decline, but positive and therefore the TP curve slopes downward.	The AP continues to decline, but still positive.	The MP continues to decline, but negative. Therefore, MP curve goes below the X-axis.