Economic Application of Matrix Algebra:

Matrix algebra is generally used in different fields of economics, such as solving market models, national income model, input-output analysis, analysis of external sector on equilibrium of an economy, analysis of equilibrium of goods market and money market, etc. Some are discussed here -

1. Equilibrium of Simple Market Model:

A market model can be solved with the help of matrix algebra. Thus equilibrium market price ($\bar{P}$) and equilibrium quantity ($\bar{Q}$) can be derived from a market model through the application of matrix algebra.

For that purpose, let us consider a simple linear market model as -

Demand function, $Q_d = a - bP$

Supply function, $Q_s = -c + dP$

Equilibrium condition $Q_d = Q_s$

where Q_d = quantity demanded, a = intercept term, b = slope parameter, P = price of the product, Q_s = quantity supplied, c = intercept term, d = slope parameter

(I)

Rewriting the above given market model as -

$$1.Q_d + 0.Q_s + bP = a$$
$$0.Q_d + 1.Q_s - dP = -c$$
$$1.Q_d - 1.Q_s + 0.P = 0$$

Now converting the above market model into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$ ——— (2)

Now applying Cramer's rule to the above matrix notation, we have

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} a & 0 & b \\ -c & 1 & -d \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{—— (3)}$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & a & b \\ 0 & -c & -d \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{—— (4)}$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & -c \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{—— (5)}$$

$$\therefore |A| = \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} + b\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0 - (-d)(-1)) - 0(0.0 - (-d).1) + b(0.(-1) - 1.1)$$

$$= 1(0 - d) - 0 + b(0 - 1)$$

$$= -d - b = -(b + d)$$

$$\therefore |A_1| = \begin{vmatrix} a & 0 & b \\ -c & 1 & -d \\ 0 & -1 & 0 \end{vmatrix} = a\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -c & -d \\ 0 & 0 \end{vmatrix} + b\begin{vmatrix} -c & 1 \\ 0 & -1 \end{vmatrix}$$

$$= a(1.0 - (-d)(-1)) - 0((-c).0 - (-d).0) + b((-c).(-1) - 1.0)$$

$$= a(0 - d) - 0 + b(c - 0)$$

$$= -ad + bc = bc - ad = -(ad - bc)$$

$$\therefore |A_2| = \begin{vmatrix} 1 & a & b \\ 0 & -c & -d \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -c & -d \\ 0 & 0 \end{vmatrix} - a\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} + b\begin{vmatrix} 0 & -c \\ 1 & 0 \end{vmatrix}$$

$$= 1((-c).0 - (-d).0) - a(0.0 - (-d).1) + b(0.0 - (-c).1)$$

$$= 1(0 + 0) - a(0 + d) + b(0 + c)$$

$$= 0 - ad + bc = bc - ad = -(ad - bc)$$

$$\therefore |A_3| = \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & -c \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -c \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -c \\ 1 & 0 \end{vmatrix} + a\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0 - (-c).(-1)) - 0(0.0 - (-c).1) + a(0.(-1) - 1.1)$$

$$= 1(0 - c) - 0(0 + c) + a(0 - 1)$$

$$= (-c - 0 - a) = -c - a = -(c + a)$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations ③, ④ and ⑤, we have -

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{-(ad-bc)}{-(b+d)} = \frac{(ad-bc)}{(b+d)}$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{-(ad-bc)}{-(b+d)} = \frac{(ad-bc)}{(b+d)}$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{-(c+a)}{-(b+d)} = \frac{(a+c)}{(b+d)}$$

Note: The matrix inversion method can also be used to solve this market model.

The simple market model is

$$\left.\begin{array}{l} \text{(a) Demand function } (Q_d) = a - bP \\ \text{(b) Supply function } (Q_s) = -c + dP \\ \text{(c) Equilibrium condition, } Q_d = Q_s \end{array}\right\} \text{——— ①}$$

Rewriting the above given market model as

$$\left.\begin{array}{l} 1.Q_d + 0.Q_s + bP = a \\ 0.Q_d + 1.Q_s - dP = -c \\ 1.Q_d - 1.Q_s + 0.P = 0 \end{array}\right\} \text{——— ②}$$

Converting the above given market model into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$ ——— ③

or $X = A^{-1}.C$ ——— ④

where, $A^{-1} = Adj(A)/|A|$

and $Adj(A) = [\text{Cofactor matrix of } A]'$

Now,

$$|A| = \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} + b\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1\left(1.0 - (-d)(-1)\right) - 0\left(0.0 - (-d).1\right) + b\left(0.(-1) - 1.1\right)$$

$$= 1(0-d) - 0(0+d) + b(0-1)$$

$$= -d - 0 - b = -b - d = -(b+d)$$

$\therefore$ Cofactor matrix of A =

$$\begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & b \\ -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & b \\ 1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(1.0-(-d)(-1)) & -(0.0-(-d).1) & +(0.(-1)-1.1) \\ -(0.0-b.(-1)) & +(1.0-b.1) & -(1.(-1)-0.1) \\ +(0.(-d)-b.1) & -(1.(-d)-b.0) & +(1.1-0.0) \end{bmatrix}$$

$$= \begin{bmatrix} (0-d) & -(0+d) & (0-1) \\ -(0+b) & (0-b) & -(-1-0) \\ (0-b) & -(-d-0) & (1-0) \end{bmatrix}$$

$$= \begin{bmatrix} -d & -d & -1 \\ -b & -b & 1 \\ -b & d & 1 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} -d & -d & -1 \\ -b & -b & 1 \\ -b & d & 1 \end{bmatrix}' = \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix}$$

Now putting the values of $|A|$, $Adj(A)$ and c in equation ④, we have —

$$X = \frac{1}{-(b+d)} \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

$$= \frac{1}{-(b+d)} \begin{bmatrix} ((-d).a + (-b).(-c) + (-b).0) \\ ((-d).a + (-b).(-c) + d.0) \\ ((-1).a) + 1.(-c) + 1.0) \end{bmatrix}$$

$$= \frac{1}{-(b+d)} \begin{bmatrix} (-ad + bc + 0) \\ (-ad + bc + 0) \\ (-a - c + 0) \end{bmatrix}$$

$$= \frac{1}{-(b+d)} \begin{bmatrix} -(ad-bc) \\ -(ad-bc) \\ -(a+c) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-(ad-bc)}{-(b+d)} \\ \frac{-(ad-bc)}{-(b+d)} \\ \frac{-(a+c)}{-(b+d)} \end{bmatrix} = \begin{bmatrix} \frac{(ad-bc)}{(b+d)} \\ \frac{(ad-bc)}{(b+d)} \\ \frac{(a+c)}{(b+d)} \end{bmatrix}$$

$$\therefore \bar{Q}_d = \frac{(ad-bc)}{(b+d)}, \quad \bar{Q}_s = \frac{(ad-bc)}{(b+d)} \text{ and } \bar{p} = \frac{(a+c)}{(b+d)}$$

Example:

1. Solve the market model by Cramer's Rule

$$Q_d = 10 - 5p$$
$$Q_s = -5 + 5p$$
$$Q_d = Q_s$$

Solution: The given market model is

$$Q_d = 10 - 5P$$
$$Q_s = -5 + 5P$$
$$Q_d = Q_s$$

The above given market model can be rewritten as

$$\left.\begin{aligned} 1.Q_d + 0.Q_s + 5P &= 10 \\ 0.Q_d + 1.Q_s - 5P &= -5 \\ 1.Q_d - 1.Q_s + 0.P &= 0 \end{aligned}\right\} \quad \text{--- (1)}$$

~~The above~~.
Now, converting the above market model into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 10 \\ -5 \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$ --- (2)

Applying Cramer's Rule to the above given matrix notation, we have -

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 10 & 0 & 5 \\ -5 & 1 & -5 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{--- (3)}$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 10 & 5 \\ 0 & -5 & -5 \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{--- (4)}$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & 10 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{--- (5)}$$

Now, $|A| = \begin{vmatrix} 1 & 0 & 5 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -5 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -5 \\ 1 & 0 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$

$$= 1(1.0-(-5).(-1)) - 0(0.0-(-5).1) + 5(0.(-1)-1.1)$$

$$= 1(0-5) - 0(0+5) + 5(0-1)$$

$$= (-5-0-5) = -10$$

$$|A_1| = \begin{vmatrix} 10 & 0 & 5 \\ -5 & 1 & -5 \\ 0 & -1 & 0 \end{vmatrix} = 10\begin{vmatrix} 1 & -5 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -5 & -5 \\ 0 & 0 \end{vmatrix} + 5\begin{vmatrix} -5 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= 10(1.0-(-5).(-1)) - 0((-5).0-(-5).0) + 5((-5).(-1)-1.0)$$

$$= 10(0-5) - 0(0+0) + 5(5-0)$$

$$= (-50-0+25) = -25$$

$$|A_2| = \begin{vmatrix} 1 & 10 & 5 \\ 0 & -5 & -5 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -5 & -5 \\ 0 & 0 \end{vmatrix} - 10\begin{vmatrix} 0 & -5 \\ 1 & 0 \end{vmatrix} + 5\begin{vmatrix} 0 & -5 \\ 1 & 0 \end{vmatrix}$$

$$= 1((-5).0-(-5).0) - 10(0.0-(-5).1) + 5(0.0-(-5).1)$$

$$= 1(0-0) - 10(0+5) + 5(0+5)$$

$$= (0-50+25) = -25$$

and, $|A_3| = \begin{vmatrix} 1 & 0 & 10 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -5 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -5 \\ 1 & 0 \end{vmatrix} + 10\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$

$$= 1(1.0-(-5)(-1)) - 0(0.0-(-5).1) + 10(0.(-1)-1.1)$$

$$= 1(0-5) - 0(0+5) + 10(0-1)$$

$$= (-5-0-10) = -15$$

Now, putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations ①, ② and ③, we have –

$\bar{Q}_d = \dfrac{|A_1|}{|A|} = \dfrac{-25}{-10} = 2.5$, and $\bar{p} = \dfrac{|A_3|}{|A|} = \dfrac{-15}{-10} = 1.5$

$\bar{Q}_s = \dfrac{|A_2|}{|A|} = \dfrac{-25}{-10} = 2.5$

2. Solve the market model by matrix inversion method -

$Q_d = 10 - 0{\cdot}4P$

$Q_s = -3 + 0{\cdot}6P$

$Q_d = Q_s$

Solution: The given market model is

$Q_d = 10 - 0{\cdot}4P$

$Q_s = -3 + 0{\cdot}6P$

$Q_d = Q_s$

The given market model can be rewritten as-

$$\left.\begin{array}{l} 1.Q_d + 0.Q_s + 0{\cdot}4P = 10 \\ 0.Q_d + 1.Q_s - 0{\cdot}6P = -3 \\ 1.Q_d - 1.Q_s + 0.P = 0 \end{array}\right\} \text{---} (1)$$

Now converting the above market model into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & 0 & 0{\cdot}4 \\ 0 & 1 & -0{\cdot}6 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 10 \\ -3 \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1}.C$ ——— (2)

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$

and, $Adj(A) = [\text{Co-factor matrix of } A]'$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0{\cdot}4 \\ 0 & 1 & -0{\cdot}6 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -0{\cdot}6 \\ (-1) & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -0{\cdot}6 \\ 1 & 0 \end{vmatrix} + 0{\cdot}4\begin{vmatrix} 0 & 1 \\ 1 & (-1) \end{vmatrix}$

$= 1(1.0 - (-0{\cdot}6)(-1)) - 0(0.0 - (-0{\cdot}6).1) + 0{\cdot}4(0(-1) - 1.1)$

$= 1(0 - 0{\cdot}6) - 0(0 + 0{\cdot}6) + 0{\cdot}4(0 - 1)$

$= (-0{\cdot}6 - 0 - 0{\cdot}4) = -1$

Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -0.6 \\ -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & -0.6 \\ 1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & 0.4 \\ -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 0.4 \\ 1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & 0.4 \\ 1 & -0.6 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 0.4 \\ 0 & -0.6 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} +(1.0-(-0.6).(-1)) & -(0.0-(-0.6).1) & +(0.(-1)-1.1) \\ -(0.0-(0.4).(-1)) & +(1.0-0.4.1) & -(1.(-1)-0.1) \\ +(0.(-0.6)-0.4.1) & -(1.(-0.6)-0.4.0) & +(1.1-0.0) \end{bmatrix}$$

$$① \longrightarrow \quad = \begin{bmatrix} (0-0.6) & -(0+0.6) & (0-1) \\ -(0+0.4) & (0-0.4) & -(1-0) \\ (0-0.4) & -(-0.6-0) & (1-0) \end{bmatrix}$$

$$= \begin{bmatrix} -0.6 & -0.6 & -1 \\ -0.4 & -0.4 & 1 \\ -0.4 & 0.6 & 1 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} -0.6 & -0.6 & -1 \\ -0.4 & -0.4 & 1 \\ -0.4 & 0.6 & 1 \end{bmatrix}' = \begin{bmatrix} -0.6 & -0.4 & -0.4 \\ -0.6 & -0.4 & 0.6 \\ -1 & 1 & 1 \end{bmatrix}$$

Now putting the values ~~in~~ of $|A|$, $Adj(A)$ and c in equation ②, we have –

$$X = \frac{1}{-1}\begin{bmatrix} -0.6 & -0.4 & -0.4 \\ -0.6 & -0.4 & 0.6 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ -3 \\ 0 \end{bmatrix} = \frac{1}{0.2}\begin{bmatrix} (-0.6).10+(-0.4).(-3)+(-0.4).0 \\ (-0.6).10+(-0.4).(-3)+0.6.0 \\ (-1).10+1.(-3)+1.0 \end{bmatrix}$$

$$= \frac{1}{-1}\begin{bmatrix} (-6+1.2-0) \\ (-6+1.2+0) \\ (-10-3+0) \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -4.8 \\ -4.8 \\ -13 \end{bmatrix} = \begin{bmatrix} \frac{-4.8}{-1} \\ \frac{-4.8}{-1} \\ \frac{-13}{-1} \end{bmatrix} = \begin{bmatrix} 4.8 \\ 4.8 \\ 13 \end{bmatrix}$$

$\therefore \bar{Q}_d = 4.8, \ \therefore \bar{Q}_s = 4.8$ and $\bar{P} = 13$

3. Solve the following market model by Cramer's rule and matrix inversion method –

$$Q_d = 50 - 2P$$
$$Q_s = -10 + 3P$$
$$Q_d = Q_s$$

Solution: The given market model is

$$Q_d = 50 - 2P$$
$$Q_s = -10 + 3P$$
$$Q_d = Q_s$$

The given market model can be rewritten as –

$$\left.\begin{aligned} 1.Q_d + 0.Q_s + 2P &= 50 \\ 0.Q_d + 1Q_s - 3P &= -10 \\ 1.Q_d - 1Q_s + 0P &= 0 \end{aligned}\right\} \text{——} (1)$$

Now, converting the given market model into matrix notation, we have –

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 50 \\ -10 \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$ ——— (2)

Applying Cramer's Rule to the above matrix notation we have –

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 50 & 0 & 2 \\ -10 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{vmatrix}} \text{——} (3)$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 50 & 2 \\ 0 & -10 & -3 \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{vmatrix}} \text{——} (4)$$

and, $\bar{P} = \dfrac{|A_3|}{|A|} = \dfrac{\begin{vmatrix} 1 & 0 & 50 \\ 0 & 1 & -10 \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{vmatrix}}$ —— (5)

$$\therefore |A| = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -3 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0-(-3).(-1)) - 0(0.0-(-3).1) + 2(0.(-1)-1.1)$$

$$= 1(0-3) - 0(0+3) + 2(0-1) = (-3-2) = -5$$

$$\therefore |A_1| = \begin{vmatrix} 50 & 0 & 2 \\ -10 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = 50\begin{vmatrix} 1 & -3 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -10 & -3 \\ 0 & 0 \end{vmatrix} + 2\begin{vmatrix} -10 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= 50(1.0-(-3).(-1)) - 0((-10).0-(-3).0) + 2((-10).(-1)-1.0)$$

$$= 50(0-3) - 0(0-0) + 2(10-0) = (-150+20) = -130$$

$$|A_2| = \begin{vmatrix} 1 & 50 & 2 \\ 0 & -10 & -3 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -10 & -3 \\ 0 & 0 \end{vmatrix} - 50\begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & -10 \\ 1 & 0 \end{vmatrix}$$

$$= 1((-10).0-(-3).0) - 50(0.0-(-3).1) + 2(0.0-(-10).1)$$

$$= 1(0-0) - 50(0+3) + 2(0+10) = (-150+20) = -130$$

and, $|A_3| = \begin{vmatrix} 1 & 0 & 50 \\ 0 & 1 & -10 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -10 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -10 \\ 1 & 0 \end{vmatrix} + 50\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$

$$= 1(1.0-(-10).(-1)) - 0(0.0-(-10).1) + 50(0.(-1)-1.1)$$

$$= 1(0-10) - 0(0+10) + 50(0-1) = (-10-50) = -60$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations (3), (4) and (5), we have –

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{-130}{-5} = 26$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{-130}{-5} = 26$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{-60}{-5} = 12$$

The above given market model can also be solved with the help of matrix inversion method.
The equation ② of this given problem is

$$AX = C$$

$$\text{or } X = A^{-1}.C \quad \text{——— ⑥}$$

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$

and $Adj(A) = [\text{Co-factor matrix of } A]'$

$$\therefore \text{Co-factor matrix of } A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -3 \\ -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(1.0-(-3).(-1)) & -(0.0-(-3).1) & +(0.(-1)-1.1) \\ -(0.0-2.(-1)) & +(1.0-2.1) & -(1.(-1)-0.1) \\ +(0.(-3)-2.1) & -(1.(-3)-2.0) & +(1.1-0.0) \end{bmatrix}$$

$$= \begin{bmatrix} (0-3) & -(0+3) & (0-1) \\ -(0+2) & (0-2) & -(-1-0) \\ (0-2) & -(-3-0) & (1-0) \end{bmatrix} = \begin{bmatrix} -3 & -3 & -1 \\ -2 & -2 & 1 \\ -2 & 3 & 1 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} -3 & -3 & -1 \\ -2 & -2 & 1 \\ -2 & 3 & 1 \end{bmatrix}' = \begin{bmatrix} -3 & -2 & -2 \\ -3 & -2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

Now putting the values of $|A|$, $Adj(A)$ and c in equation ⑥, we have -

$$X = \frac{1}{-5}\begin{bmatrix} -3 & -2 & -2 \\ -3 & -2 & 3 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 50 \\ -10 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} ((-3).50 + (-2).(-10) + (-2).0) \\ ((-3).50 + (-2).(-10) + 3.0) \\ ((-1).50 + 1.(-10) + 1.0) \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} (-150 + 20 - 0) \\ (-150 + 20 + 0) \\ (-50 - 10 + 0) \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} -130 \\ -130 \\ -60 \end{bmatrix} = \begin{bmatrix} \frac{-130}{-5} \\ \frac{-130}{-5} \\ \frac{-60}{-5} \end{bmatrix} = \begin{bmatrix} 26 \\ 26 \\ 12 \end{bmatrix}$$

$\therefore \bar{Q}_d = 26, \quad \bar{Q}_s = 26$ and $\bar{P} = 12$

2. Simple National Income Model: A simple linear national income model can also be solved by using matrix algebra. Thus, equlibrium level of national income ($\bar{Y}$), consumption level ($\bar{C}$) and total tax callection ($\bar{T}$) can be derived from a simple linear national income model by using Cramer's Rule ~~or~~ and Matrix inversion method -

(a) **Application of Cramer's Rule:** Let us consider a simple linear national income model as -

$$\left.\begin{aligned} Y &= C + I_0 + G_0 \\ C &= \alpha + \beta(Y - T) \quad (1 > \beta > 0) \\ T &= \delta Y \quad (1 > \delta > 0) \end{aligned}\right\} \text{①}$$

where, Y = national income, C = consumption expenditure, I_0 = autonomous investment, G_0 = government expenditures, T = Total Tax, β = Marginal propensity to consume, δ = rate of income tax, α = level of consumption when disposable income $(Y-T)$ is zero.

The given national income model can be rewritten as -

$$\left.\begin{aligned} 1.Y - 1.C + 0.T &= (I_0 + G_0) \\ -\beta Y + 1.C + \beta.T &= \alpha \\ -\delta Y + 0.C + 1.T &= 0 \end{aligned}\right\} \quad \text{——} \ (2)$$

Now converting the above national income model into matrix form or notation as —

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} (I_0 + G_0) \\ \alpha \\ 0 \end{bmatrix}}_{C}$$

$$\text{or} \quad AX = C \quad \text{——} \ (3)$$

Applying Cramer's rule to the above given matrix notation we have -

$$\bar{Y} = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} (I_0+G_0) & -1 & 0 \\ \alpha & 1 & \beta \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix}} \quad \text{——} \ (4)$$

$$\bar{C} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & (I_0+G_0) & 0 \\ -\beta & \alpha & \beta \\ -\delta & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix}} \quad \text{——} \ (5)$$

$$\bar{T} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & (I_0+G_0) \\ -\beta & 1 & \alpha \\ -\delta & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix}} \quad \text{——} \ (6)$$

$$\therefore |A| = \begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & \beta \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -\beta & \beta \\ -\delta & 1 \end{vmatrix} + 0\begin{vmatrix} -\beta & 1 \\ -\delta & 0 \end{vmatrix}$$

$$= 1(1.1 - \beta.0) + 1((-\beta).1 - \beta.(-\delta)) + 0((-\beta).0 - 1.(-\delta))$$

$$= 1(1-0) + 1(-\beta + \beta\delta) + 0(0+\delta)$$

$$= (1 - \beta + \beta\delta + 0) = (1 - \beta(1-\delta))$$

$$|A_1| = \begin{vmatrix} (I_0+G_0) & -1 & 0 \\ \alpha & 1 & \beta \\ 0 & 0 & 1 \end{vmatrix} = (I_0+G_0)\begin{vmatrix} 1 & \beta \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} \alpha & \beta \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} \alpha & 1 \\ 0 & 0 \end{vmatrix}$$

$$= (I_0+G_0)(1.1 - \beta.0) + 1(\alpha.1 - \beta.0) + 0(\alpha.0 - 1.0)$$

$$= (I_0+G_0)(1-0) + 1(\alpha - 0) + 0(0-0)$$

$$= (I_0 + G_0 + \alpha)$$

$$|A_2| = \begin{vmatrix} 1 & (I_0+G_0) & 0 \\ -\beta & \alpha & \beta \\ -\delta & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} \alpha & \beta \\ 0 & 1 \end{vmatrix} - (I_0+G_0)\begin{vmatrix} -\beta & \beta \\ -\delta & 1 \end{vmatrix} + 0\begin{vmatrix} -\beta & \alpha \\ -\delta & 0 \end{vmatrix}$$

$$= 1(\alpha.1 - \beta.0) - (I_0+G_0)((-\beta).1 - \beta.(-\delta)) + 0((-\beta).0 - \alpha.(-\delta))$$

$$= 1(\alpha - 0) - (I_0+G_0)(-\beta + \beta\delta) + 0(0 + \alpha\delta)$$

(3) $$= \alpha - (I_0+G_0)(-\beta)(1-\delta) + 0$$

$$= \alpha + \beta(1-\delta)(I_0+G_0)$$

and $$|A_3| = \begin{vmatrix} 1 & -1 & (I_0+G_0) \\ -\beta & 1 & \alpha \\ -\delta & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & \alpha \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} -\beta & \alpha \\ -\delta & 0 \end{vmatrix} + (I_0+G_0)\begin{vmatrix} -\beta & 1 \\ -\delta & 0 \end{vmatrix}$$

$$= 1(1.0 - \alpha.0) + 1((-\beta).0 - \alpha.(-\delta)) + (I_0+G_0)((-\beta).0 - 1.(-\delta))$$

$$= 1(0-0) + 1(0 + \alpha\delta) + (I_0+G_0)(0+\delta)$$

$$= (0 + \alpha\delta + \delta(I_0+G_0)) = \delta(\alpha + I_0 + G_0)$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equation (4), (5) and (6), we have

$$\bar{Y} = \frac{|A_1|}{|A|} = \frac{(I_0 + G_0 + \alpha)}{(1-\beta(1-\delta))}$$

$$\bar{C} = \frac{|A_2|}{|A|} = \frac{(\alpha + \beta(1-\delta)(I_0+G_0))}{(1-\beta(1-\delta))}$$

$$\bar{T} = \frac{|A_3|}{|A|} = \frac{\delta(\alpha + I_0 + G_0)}{(1-\beta(1-\delta))}$$

~~Notes~~

(b) Application of Matrix inversion method:

The simple linear national income model is -

$$\left.\begin{aligned} Y &= C + I_0 + G_0 \\ C &= \alpha + \beta(Y - T) \\ T &= \delta Y \end{aligned}\right\} \text{——— (1)}$$

The above given national income model can be rewritten as -

$$\left.\begin{aligned} 1.Y - 1.C + 0.T &= (I_0 + G_0) \\ -\beta Y + 1.C + \beta T &= \alpha \\ -\delta Y + 0.C + 1.T &= 0 \end{aligned}\right\} \text{——— (2)}$$

Now converting the above national income model into matrix form or notation as -

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} (I_0 + G_0) \\ \alpha \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$

$\Rightarrow X = A^{-1}.C$ ——— (3)

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$

and $Adj(A) = [\text{Co-factor matrix of } A]'$

As we have already derived that $|A| = (1-\beta(1-\delta))$

co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & \beta \\ 0 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -\beta & \beta \\ -\delta & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -\beta & 1 \\ -\delta & 0 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ -\delta & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -\delta & 0 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 1 & \beta \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ -\beta & \beta \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -\beta & 1 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} +(1.1-\beta.0) & -((-\beta).1-\beta.(-\delta)) & +((-\beta).0-1.(-\delta)) \\ -((-1).1-0.0) & +(1.1-0.(-\delta)) & -(1.0-(-1)(-\delta)) \\ +((-1).\beta-0.1) & -(1.\beta-0.(-\beta)) & +(1.1-(-1).(-\beta)) \end{bmatrix}$$

$$= \begin{bmatrix} (1-0) & -(-\beta+\beta\delta) & (0+\delta) \\ -(-1-0) & (1+0) & -(0-\delta) \\ (-\beta-0) & -(\beta+0) & (1-\beta) \end{bmatrix} = \begin{bmatrix} 1 & \beta(1-\delta) & \delta \\ 1 & 1 & \delta \\ -\beta & -\beta & (1-\beta) \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} 1 & \beta(1-\delta) & \delta \\ 1 & 1 & \delta \\ -\beta & -\beta & (1-\beta) \end{bmatrix}' = \begin{bmatrix} 1 & 1 & -\beta \\ \beta(1-\delta) & 1 & -\beta \\ \delta & \delta & (1-\beta) \end{bmatrix}$$

Now putting the values of $|A|$, $Adj(A)$ and C in equation ③, we have -

$$X = \frac{1}{(1-\beta(1-\delta))} \begin{bmatrix} 1 & 1 & -\beta \\ \beta(1-\delta) & 1 & -\beta \\ \delta & \delta & (1-\beta) \end{bmatrix} \begin{bmatrix} (I_0+G_0) \\ \alpha \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \frac{1}{(1-\beta(1-\delta))} \begin{bmatrix} (1.(I_0+G_0)+1.\alpha+(-\beta).0) \\ (\beta.(1-\delta)(I_0+G_0)+1.\alpha+(-\beta).0) \\ (\delta.(I_0+G_0)+\delta.\alpha+(1-\beta).0) \end{bmatrix}$$

$$= \frac{1}{(1-\beta(1-\delta))} \begin{bmatrix} (I_0+G_0+\alpha) \\ (\alpha+\beta(1-\delta)(I_0+G_0)) \\ (\delta(I_0+G_0)+\delta.\alpha+0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(I_0+G_0+\alpha)}{(1-\beta(1-\delta))} \\ \frac{(\alpha+\beta(1-\delta)(I_0+G_0))}{(1-\beta(1-\delta))} \\ \frac{\delta(\alpha+I_0+G_0)}{(1-\beta(1-\delta))} \end{bmatrix}.$$

$$\therefore \bar{Y} = \frac{(I_0+G_0+\alpha)}{(1-\beta(1-\delta))}, \ \bar{C} = \frac{(\alpha+\beta(1-\delta)(I_0+G_0))}{(1-\beta(1-\delta))} \ \& \ \bar{T} = \frac{\delta(\alpha+I_0+G_0)}{(1-\beta(1-\delta))}$$

Example:

1. Solve the following national income model by Cramer's Rule.

~~Solution:~~ $Y = C + I_0 + G_0$

$C = 120 + 0.4(Y-T)$

$T = 0.2Y$

where, $I_0 = 1800$ and $G_0 = 800$

Solution: The given national income model is

$Y = C + I_0 + G_0$

$C = 120 + 0.4(Y-T)$

$T = 0.2Y$, where $I_0 = 1800$, $G_0 = 800$

$$\text{or} \quad \left.\begin{aligned} Y &= C + 1800 + 800 \\ C &= 120 + 0.4(Y-T) \\ T &= 0.2Y \end{aligned}\right\} \text{—— (1)}$$

The above given national income model can be rewritten as –

$$\left.\begin{array}{l} 1.Y - 1.C + 0.T = 2600 \\ -0.4Y + 1.C + 0.4.T = 120 \\ -0.2Y + 0.C + 1.T = 0 \end{array}\right\} \quad \text{—— (2)}$$

Now converting the above given national income model into matrix form or notation as

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -0.4 & 1 & 0.4 \\ -0.2 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2600 \\ 120 \\ 0 \end{bmatrix}}_{C}$$

$$\text{or} \quad AX = C \quad \text{—— (3)}$$

Applying Cramer's Rule to the above matrix notaion we have –

$$\bar{Y} = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 2600 & -1 & 0 \\ 120 & 1 & 0.4 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -0.4 & 1 & 0.4 \\ -0.2 & 0 & 1 \end{vmatrix}} \quad \text{—— (4)}$$

$$\bar{C} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 2600 & 0 \\ -0.4 & 120 & 0.4 \\ -0.2 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -0.4 & 1 & 0.4 \\ -0.2 & 0 & 1 \end{vmatrix}} \quad \text{—— (5)}$$

$$\bar{T} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & 2600 \\ -0.4 & 1 & 120 \\ -0.2 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -0.4 & 1 & 0.4 \\ -0.2 & 0 & 1 \end{vmatrix}} \quad \text{—— (6)}$$

$$\therefore |A| = \begin{vmatrix} 1 & -1 & 0 \\ -0.4 & 1 & 0.4 \\ -0.2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0.4 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -0.4 & 0.4 \\ -0.2 & 1 \end{vmatrix} + 0\begin{vmatrix} -0.4 & 1 \\ -0.2 & 0 \end{vmatrix}$$

$$= 1(1.1 - 0.4.0) + 1((-0.4).1 - 0.4.(-0.2)) + 0((-0.4).0 - 1.(-0.2))$$

$$= 1(1-0) + 1(-0.4 + 0.08) + 0(0 + 0.2)$$

$$= (1 - 0.4 + 0.08 + 0) = 0.64$$

$$|A_1| = \begin{vmatrix} 2600 & -1 & 0 \\ 120 & 1 & 0.4 \\ 0 & 0 & 1 \end{vmatrix} = 2600\begin{vmatrix} 1 & 0.4 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 120 & 0.4 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 120 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 2600(1.1 - 0.4.0) + 1(120.1 - 0.4.0) + 0(120.0 - 1.0)$$

$$= 2600(1-0) + 1(120-0) + 0(0-0)$$

$$= (2600 + 120 + 0) = 2720$$

$$|A_2| = \begin{vmatrix} 1 & 2600 & 0 \\ -0.4 & 120 & 0.4 \\ -0.2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 120 & 0.4 \\ 0 & 1 \end{vmatrix} - 2600\begin{vmatrix} -0.4 & 0.4 \\ -0.2 & 1 \end{vmatrix} + 0\begin{vmatrix} -0.4 & 120 \\ -0.2 & 0 \end{vmatrix}$$

$$= 1(120.1 - 0.4.0) - 2600((-0.4).1 - 0.4.(-0.2)) + 0((-0.4).0 - 120.(-0.2))$$

$$= 1(120-0) - 2600(-0.4 + 0.08) + 0(0 + 24)$$

$$= (120 + 1040 - 208 + 0) = 952$$

and, $$|A_3| = \begin{vmatrix} 1 & -1 & 2600 \\ -0.4 & 1 & 120 \\ -0.2 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 120 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} -0.4 & 120 \\ -0.2 & 0 \end{vmatrix} + 2600\begin{vmatrix} -0.4 & 1 \\ -0.2 & 0 \end{vmatrix}$$

$$= 1(1.0 - 120.0) + 1((-0.4).0 - 120.(-0.2)) + 2600((-0.4).0 - 1.(-0.2))$$

$$= 1(0-0) + 1(0+24) + 2600(0+0.2)$$

$$= (0 + 24 + 520) = 544$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations ④, ⑤ and ⑥, we have –

$$\bar{Y} = \frac{|A_1|}{|A|} = \frac{2720}{0.64} = 4250$$

$$\bar{C} = \frac{|A_2|}{|A|} = \frac{952}{0.64} = 1,487.5$$

and $$\bar{T} = \frac{|A_3|}{|A|} = \frac{544}{0.64} = 850$$

2. Solve the following national income model by matrix inversion method.

$$Y = C + I_0 + G_0$$
$$C = 140 + 0.6(Y - T)$$
$$T = 0.4Y \quad , \text{ where } I_0 = 1000 \text{ and } G_0 = 750$$

Solution: The given national income model is

$$\left.\begin{aligned} Y &= C + I_0 + G_0 \\ C &= 140 + 0.6(Y-T) \\ T &= 0.4Y, \text{ where } I_0 = 1000 \text{ and } G_0 = 750 \end{aligned}\right\} \text{—} (1)$$

The given national income model can be rewritten as -

$$\left.\begin{aligned} 1.Y - 1.C + 0.T &= 1000 + 750 \\ -0.6Y + 1.C + 0.6T &= 140 \\ -0.4Y + 0.C + 1.T &= 0 \end{aligned}\right\} \text{—} (2)$$

The above given national income model is converted into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -0.6 & 1 & 0.6 \\ -0.4 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1750 \\ 140 \\ 0 \end{bmatrix}}_{C} \text{———} (3)$$

or $\quad AX = C$

or $\quad X = A^{-1}.C$ ——— (4)

We know that, $A^{-1} = \dfrac{Adj(A)}{|A|}$, and

$$Adj(A) = [\text{co-factor matrix of } A]'$$

$$\therefore |A| = \begin{vmatrix} 1 & -1 & 0 \\ -0.6 & 1 & 0.6 \\ -0.4 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0.6 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -0.6 & 0.6 \\ -0.4 & 1 \end{vmatrix} + 0\begin{vmatrix} -0.6 & 1 \\ -0.4 & 0 \end{vmatrix}$$

$$= 1(1.1 - 0.6.0) + 1((-0.6).1 - 0.6.(-0.4)) + 0((-0.6).0 - 1(-0.4))$$
$$= 1(1-0) + 1(-0.6 + 0.24) + 0(0 + 0.4)$$
$$= (1 - 0.36 + 0) = 0.64$$

$$\therefore \text{Co-factor of } A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 0.6 \\ 0 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -0.6 & 0.6 \\ -0.4 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -0.6 & 1 \\ -0.4 & 0 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ -0.4 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -0.4 & 0 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 1 & 0.6 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ -0.6 & 0.6 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -0.6 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(1.1-0.6.0) & -((-0.6).1-0.6.(-0.4)) & +((-0.6).0-1.(-0.4)) \\ -((-1).1-0.0) & +(1.1-0.(-0.4)) & -(1.0-(-1).(-0.4)) \\ +((-1).0.6-0.1) & -(1.0.6-0.(-0.6)) & +(1.1-(-1).(-0.6)) \end{bmatrix}$$

$$= \begin{bmatrix} +(1-0) & -(-0.6+0.24) & +(0+0.4) \\ -(-1-0) & +(1-0) & -(0-0.4) \\ +(-0.6-0) & -(0.6+0) & +(1-0.6) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.36 & 0.4 \\ 1 & 1 & 0.4 \\ -0.6 & -0.6 & 0.4 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} 1 & 0.36 & 0.4 \\ 1 & 1 & 0.4 \\ -0.6 & -0.6 & 0.4 \end{bmatrix}' = \begin{bmatrix} 1 & 1 & -0.6 \\ 0.36 & 1 & -0.6 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$$

Now, ~~applyin~~ putting the values of $|A|$, $Adj(A)$ and C in equation ④, we have -

$$X = \frac{1}{0.64}\begin{bmatrix} 1 & 1 & -0.6 \\ 0.36 & 1 & -0.6 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} 1750 \\ 140 \\ 0 \end{bmatrix}$$

$$= \frac{1}{0.64}\begin{bmatrix} (1.1750+1.140+(-0.6).0) \\ (0.36.1750+1.140+(-0.6).0) \\ (0.4.1750+0.4.140+0.4.0) \end{bmatrix} = \frac{1}{0.64}\begin{bmatrix} (1750+140+0) \\ (630+140+0) \\ (700+56+0) \end{bmatrix}$$

$$= \frac{1}{0.64}\begin{bmatrix}1890\\770\\756\end{bmatrix} = \begin{bmatrix}\frac{1890}{0.64}\\ \frac{770}{0.64}\\ \frac{756}{0.64}\end{bmatrix} = \begin{bmatrix}2953.125\\1203.125\\1181.25\end{bmatrix}$$

$\therefore \bar{Y} = 2953.12$, $\bar{C} = 1203.12$ and $\bar{T} = 1181.25$

3. Solve the following national income model by Cramer's Rule –

$$Y = C + I_0$$
$$C = 200 + 0.8(Y - T)$$
$$T = 40 + 0.2Y \qquad \text{where } I_0 = 800$$

Solution: The given national income model is

$$\left.\begin{aligned} Y &= C + I_0 \\ C &= 200 + 0.8(Y-T) \\ T &= 40 + 0.2Y, \text{ where } I_0 = 800 \end{aligned}\right\} \text{—— (1)}$$

The given national income model can be rewritten as –

$$\left.\begin{aligned} 1.Y - 1C + 0.T &= 800 \\ -0.8Y + 1.C + 0.8T &= 200 \\ -0.2Y + 0.C + 1.T &= 40 \end{aligned}\right\} \text{—— (2)}$$

Converting the above given national income model into matrix form or notation as –

$$\underbrace{\begin{bmatrix}1 & -1 & 0\\ -0.8 & 1 & 0.8\\ -0.2 & 0 & 1\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}Y\\C\\T\end{bmatrix}}_{X} = \underbrace{\begin{bmatrix}800\\200\\40\end{bmatrix}}_{C}$$

or $AX = C$ —— (3)

Applying Cramer's Rule to the above given matrix notation, we have –

$$\bar{Y} = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix}800 & -1 & 0\\ 200 & 1 & 0.8\\ 40 & 0 & 1\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ -0.8 & 1 & 0.8\\ -0.2 & 0 & 1\end{vmatrix}} \text{ — (4)}, \quad \bar{C} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix}1 & 800 & 0\\ -0.8 & 200 & 0.8\\ -0.2 & 40 & 1\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ -0.8 & 1 & 0.8\\ -0.2 & 0 & 1\end{vmatrix}} \text{ — (5)}$$

$$T = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & 800 \\ -0.8 & 1 & 200 \\ -0.2 & 0 & 40 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -0.8 & 1 & 0.8 \\ -0.2 & 0 & 1 \end{vmatrix}} \quad \text{--- (6)}$$

Now, $|A| = \begin{vmatrix} 1 & -1 & 0 \\ -0.8 & 1 & 0.8 \\ -0.2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0.8 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -0.8 & 0.8 \\ -0.2 & 1 \end{vmatrix} + 0\begin{vmatrix} -0.8 & 1 \\ -0.2 & 0 \end{vmatrix}$

$= 1(1.1 - 0.8.0) + 1((-0.8).1 - 0.8.(-0.2)) + 0((-0.8).0 - 1.(-0.2))$

$= 1(1-0) + 1(-0.8 + 0.16) + 0$

$= (1 - 0.8 + 0.16) = 0.36$

$|A_1| = \begin{vmatrix} 800 & -1 & 0 \\ 200 & 1 & 0.8 \\ 40 & 0 & 1 \end{vmatrix} = 800\begin{vmatrix} 1 & 0.8 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 200 & 0.8 \\ 40 & 1 \end{vmatrix} + 0\begin{vmatrix} 200 & 1 \\ 40 & 0 \end{vmatrix}$

$= 800(1.1 - 0.8.0) + 1(200.1 - 0.8.40) + 0(200.0 - 1.40)$

$= 800(1-0) + 1(200 - 32) + 0$

$= (800 + 200 - 32) = 968$

$|A_2| = \begin{vmatrix} 1 & 800 & 0 \\ -0.8 & 200 & 0.8 \\ -0.2 & 40 & 1 \end{vmatrix} = 1\begin{vmatrix} 200 & 0.8 \\ 40 & 1 \end{vmatrix} - 800\begin{vmatrix} -0.8 & 0.8 \\ -0.2 & 1 \end{vmatrix} + 0\begin{vmatrix} -0.8 & 200 \\ -0.2 & 40 \end{vmatrix}$

$= 1(200.1 - 0.8.40) - 800((-0.8).1 - 0.8.(-0.2)) + 0((-0.8).40 - 200.(-0.2))$

$= 1(200 - 32) - 800(-0.8 + 0.16) + 0$

$= (168 + 512) = 680$

and $|A_3| = \begin{vmatrix} 1 & -1 & 800 \\ -0.8 & 1 & 200 \\ -0.2 & 0 & 40 \end{vmatrix} = 1\begin{vmatrix} 1 & 200 \\ 0 & 40 \end{vmatrix} - (-1)\begin{vmatrix} -0.8 & 200 \\ -0.2 & 40 \end{vmatrix} + 800\begin{vmatrix} -0.8 & 1 \\ -0.2 & 0 \end{vmatrix}$

$= 1(1.40 - 200.0) + 1((-0.8).40 - 200.(-0.2)) + 800((-0.8).0 - 1.(-0.2))$

$= 1(40 - 0) + 1(-32 + 40) + 800(0 + 0.2)$

$= (40 - 32 + 40 + 160) = 208$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations ④, ⑤ and ⑥, we have –

$$\bar{Y} = \frac{|A_1|}{|A|} = \frac{968}{0.36} = 2,688.89$$

$$\bar{C} = \frac{|A_2|}{|A|} = \frac{680}{0.36} = 1,888.89$$

$$\bar{T} = \frac{|A_3|}{|A|} = \frac{208}{0.36} = 577.78$$

4. Solve the following national income model by matrix inversion method –

$$Y = C + I_0 + G_0$$
$$C = 200 + 0.8(Y - T)$$
$$T = 50 + 0.3Y, \text{ where } I_0 = 500 \text{ and } G_0 = 400$$

Solution: The given national income model is

$$\left.\begin{aligned} Y &= C + I_0 + G_0 \\ C &= 200 + 0.8(Y - T) \\ T &= 50 + 0.3Y, \text{ where } I_0 = 500 \text{ and } G_0 = 400 \end{aligned}\right\} \text{— ①}$$

The above given national income model can be rewritten as –

$$\left.\begin{aligned} 1.Y - 1.C + 0.T &= 500 + 400 \\ -0.8Y + 1.C + 0.8T &= 200 \\ -0.3Y + 0.C + 1.T &= 50 \end{aligned}\right\} \text{— ②}$$

Converting the above given national income model into matrix form or notation, we have –

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -0.8 & 1 & 0.8 \\ -0.3 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 900 \\ 200 \\ 50 \end{bmatrix}}_{C}$$

or $AX = C$

$$\Rightarrow X = A^{-1}.C \quad \text{— ③}$$

We know that, $A^{-1} = Adj(A)/|A|$, and

$$Adj(A) = [\text{co-factor matrix of } A]'$$

Now,
$$|A| = \begin{vmatrix} 1 & -1 & 0 \\ -0.8 & 1 & 0.8 \\ -0.3 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0.8 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -0.8 & 0.8 \\ -0.3 & 1 \end{vmatrix} + 0\begin{vmatrix} -0.8 & 1 \\ -0.3 & 0 \end{vmatrix}$$

$$= 1(1.1 - 0.8.0) + 1((-0.8).1 - 0.8.(-0.3)) + 0((-0.8).0 - 1.(-0.3))$$

$$= 1(1-0) + 1(-0.8 + 0.24) + 0(0 + 0.3)$$

$$= (1 - 0.8 + 0.24 + 0) = 0.44$$

$$\text{Co-factor matrix of } A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 0.8 \\ 0 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -0.8 & 0.8 \\ -0.3 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -0.8 & 1 \\ -0.3 & 0 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ -0.3 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -0.3 & 0 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 1 & 0.8 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ -0.8 & 0.8 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -0.8 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(1.1 - 0.8.0) & -((-0.8).1 - 0.8.(-0.3)) & +((-0.8).0 - 1.(-0.3)) \\ -((-1).1 - 0.0) & +(1.1 - (0.(-0.3))) & +(1.0 - (-1).(-0.3)) \\ +((-1).0.8 - 0.1) & -(1.0.8 - 0.(-0.8)) & +(1.1 - (-1).(-0.8)) \end{bmatrix}$$

$$= \begin{bmatrix} (1-0) & -(-0.8 + 0.24) & (0 + 0.3) \\ -(-1-0) & (1+0) & (0 - 0.3) \\ (-0.8-0) & -(0.8-0) & (1 - 0.8) \end{bmatrix} = \begin{bmatrix} 1 & 0.56 & 0.3 \\ 1 & 1 & -0.3 \\ -0.8 & -0.8 & 0.2 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} 1 & 0.56 & 0.3 \\ 1 & 1 & -0.3 \\ -0.8 & -0.8 & 0.2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 & -0.8 \\ 0.56 & 1 & -0.8 \\ 0.3 & -0.3 & 0.2 \end{bmatrix}$$

Now putting the values of $|A|$, $Adj(A)$ and c in equation ③, we have -

$$X = \frac{1}{0.44}\begin{bmatrix} 1 & 1 & -0.8 \\ 0.56 & 1 & -0.8 \\ 0.3 & -0.3 & 0.2 \end{bmatrix}\begin{bmatrix} 900 \\ 200 \\ 50 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{T} \end{bmatrix} = \frac{1}{0.44}\begin{bmatrix} (1.900 + 1.200 + (-0.8).50) \\ (0.56.900 + 1.200 + (-0.8).50) \\ (0.3.900 + (-0.3).200 + 0.2.50) \end{bmatrix}$$

$$= \frac{1}{0.44}\begin{bmatrix} (900 + 200 - 40) \\ (504 + 200 - 40) \\ (270 - 60 + 10) \end{bmatrix} = \frac{1}{0.44}\begin{bmatrix} 1060 \\ 664 \\ 220 \end{bmatrix} = \begin{bmatrix} \frac{1060}{0.44} \\ \frac{664}{0.44} \\ \frac{220}{0.44} \end{bmatrix} = \begin{bmatrix} 2,409.09 \\ 1,509.09 \\ 500 \end{bmatrix}$$

$\therefore \bar{Y} = 2409.09$, $\bar{C} = 1,509.09$ and $\bar{T} = 500$

3. Equilibrium of a Market Model with Indirect Taxes:

In a competitive market, equilibrium price and quantity of a product are determined by demand (Q_d) and supply (Q_s) of the product. If an indirect tax - either excise duty or sales tax is introduced on the product, then equilibrium price and quantity are affected. An excise tax or a sales tax is paid by the seller or producer at first and then the burden of tax is shifted to the consumer.

So, in order to analyse the effects of an excise duty in a competitive market, let us assume an excise duty (i.e., 't', a fixed amount) per unit of output is imposed on the supplier. Therefore, the price received by the supplier (p^*) is less than the market price (P), and it will be

$$p^* = P - t \quad , \text{ where } 1 > t > 0$$

Now, assuming a market model with introduction of an exice duty 't' per unit of output as -

$$\left.\begin{array}{l} \text{Demand function, } Q_d = a - bP \quad (a, b > 0) \\ \text{Supply function, } Q_s = -c + d(P - t) \quad (c, d > 0) \\ \text{Equilibrium function, } Q_d = Q_s \end{array}\right\} \text{——— } ①$$

The above give market model can be rewritten as-

$$\left.\begin{aligned} 1.Q_d + 0.Q_s + bP &= a \\ 0.Q_d + 1.Q_s - dP &= -c - dt \\ 1.Q_d - 1.Q_s + 0.P &= 0 \end{aligned}\right\} \text{——— (2)}$$

Now, converting the above given market model into matrix form or notation, we have—

$$\therefore \underbrace{\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ -(c+dt) \\ 0 \end{bmatrix}}_{C}$$

$$\text{or } AX = C \text{ ——— (3)}$$

Applying Cramer's Rule to the above given matrix notation, we have -

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} a & 0 & b \\ -(c+dt) & 1 & -d \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} \text{ ——— (4)}$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & a & b \\ 0 & -(c+dt) & -d \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} \text{ — (5)}$$

$$\text{and } \bar{P} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & -(c+dt) \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} \text{ — (6)}$$

$$\therefore |A| = \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} + b\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0 - (-d).(-1)) - 0(0.0 - (-d).1) + b(0.(-1) - 1.1)$$

$$= 1(0 - d) - 0(0 + d) + b(0 - 1)$$

$$= (-d - 0 - b) = -(b+d)$$

$$|A_1| = \begin{vmatrix} a & 0 & b \\ -(c+dt) & 1 & -d \\ 0 & -1 & 0 \end{vmatrix} = a\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -(c+dt) & -d \\ 0 & 0 \end{vmatrix} + b\begin{vmatrix} -(c+dt) & 1 \\ 0 & -1 \end{vmatrix}$$

$$= a\left(1.0-(-d).(-1)\right) - 0 + b\left((-(c+dt)).(-1) - 1.0\right)$$

$$= a(0-d) + b\left((c+dt)-0\right)$$

$$= (-ad + bc + bdt) = -\left[(ad-bc) - bdt\right]$$

$$|A_2| = \begin{vmatrix} 1 & a & b \\ 0 & -(c+dt) & -d \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -(c+dt) & -d \\ 0 & 0 \end{vmatrix} - a\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} + b\begin{vmatrix} 0 & -(c+dt) \\ 1 & 0 \end{vmatrix}$$

$$= 1\left(\{-(c+dt).0 - (-d).0\}\right) - a\left(0.0 - (-d).1\right) + b\{0.0 - (-(c+dt)).1\}$$

$$= 1(0+0) - a(0+d) + b\{0 + (c+dt)\}$$

$$= (0 - ad + bc + bdt) = -\left[(ad-bc) - bdt\right]$$

$$|A_3| = \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & -(c+dt) \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -(c+dt) \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -(c+dt) \\ 1 & 0 \end{vmatrix} + a\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1\left(1.0 - (-(c+dt)).(-1)\right) - 0 + a\left(0.(-1) - 1.1\right)$$

$$= 1\left(0 - (c+dt)\right) + a(0-1)$$

$$= -c - dt - a$$

$$= -\left[(a+c) + dt\right]$$

- Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations (4), (5) and (6), we have —

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{-\left[(ad-bc) - bdt\right]}{-(b+d)} = \frac{(ad-bc)}{(b+d)} - \frac{bd}{(b+d)}t$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{-\left[(ad-bc) - bdt\right]}{-(b+d)} = \frac{(ad-bc)}{(b+d)} - \frac{bd}{(b+d)}t$$

and, $$\bar{P} = \frac{|A_3|}{|A|} = \frac{-\left[(a+c) + dt\right]}{-(b+d)} = \frac{(a+c)}{(b+d)} + \frac{d}{(b+d)}t$$

Note: The above market model can also be solved with the help of matrix inversion method.

The equation ③ of the above market model is

$$AX = C$$

$$\text{or } X = A^{-1}.C \quad \text{———— (7)}$$

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$, and

$$Adj\, A = [\text{co-factor matrix of } A]'$$

It is already derived that $|A| = -(b+d)$

Now, Co factor matrix of A $= \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{1+1}\begin{vmatrix} 0 & b \\ -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & b \\ 1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} +(1.0-(-d).(-1)) & -(0.0-(-d).1) & +(0.(-1)-1.1) \\ -(0.0-b.(-1)) & +(1.0-b.1) & -(1.(-1)-0.1) \\ +(0.(-d)-b.1) & -(1.(-d)-b.0) & +(1.1-0.0) \end{bmatrix}$$

$$= \begin{bmatrix} (0-d) & -(0+d) & (0-1) \\ -(0+b) & (0-b) & -(-1-0) \\ (0-b) & -(-d-0) & (1-0) \end{bmatrix} = \begin{bmatrix} -d & -d & -1 \\ -b & -b & 1 \\ -b & d & 1 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} -d & -d & -1 \\ -b & -b & 1 \\ -b & d & 1 \end{bmatrix}' = \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix}$$

Now, putting the values of $|A|$, $Adj(A)$ and c in equation (7), we have —

$$X = \frac{1}{-(b+d)} \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -(c+dt) \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} Q_d \\ Q_s \\ p \end{bmatrix} = \frac{1}{-(b+d)} \begin{bmatrix} (-d).a + (-b).(-(c+dt)) + (-b).0 \\ (-d).a + (-b).(-(c+dt)) + d.0 \\ (-1).a + 1.(-(c+dt)) + 1.0 \end{bmatrix}$$

$$= \frac{1}{-(b+d)} \begin{bmatrix} (-ad + bc + bdt - 0) \\ (-ad + bc + bdt + 0) \\ (-a - c - dt + 0) \end{bmatrix} = \begin{bmatrix} \frac{-[(ad-bc) - bdt]}{-(b+d)} \\ \frac{-[(ad-bc) - bdt]}{-(b+d)} \\ \frac{-[(a+c) + dt]}{-(b+d)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(ad-bc)}{(b+d)} - \frac{bd}{(b+d)}t \\ \frac{(ad-bc)}{(b+d)} - \frac{bd}{(b+d)}t \\ \frac{(a+c)}{(b+d)} + \frac{d}{(b+d)}t \end{bmatrix}$$

$$\therefore \quad \bar{Q}_d = \frac{(ad-bc)}{(b+d)} - \frac{bd}{(b+d)}t$$

$$\bar{Q}_s = \frac{(ad-bc)}{(b+d)} - \frac{bd}{(b+d)}t$$

and $$\bar{p} = \frac{(a+c)}{(b+d)} + \frac{d}{(b+d)}t$$

Example:

1. Solve the following market model, if the government imposes an excise duty at the rate of 2 per unit of output sold. ~~Also find~~

$$Q_d = 20 - 2P$$
$$Q_s = 2P - 5$$
$$Q_d = Q_s$$

Solution: The given market model is

$$\left.\begin{aligned} Q_d &= 20 - 2P \\ Q_s &= 2P - 5 \\ Q_d &= Q_s \end{aligned}\right\} \quad -(1)$$

and it is also given that Government has introduced an excise duty at the rate of 2 per unit of output sold.

The above given market model can be rewritten as after imposition of excise duty of 2 per unit of output-

$$Q_d = 20 - 2P$$
$$Q_s = -5 + 2(P-2)$$
$$Q_d = Q_s$$

$$\text{or,}\quad \left.\begin{aligned} 1.Q_d + 0.Q_s + 2P &= 20 \\ 0.Q_d + 1.Q_s - 2P &= -9 \\ 1.Q_d - 1.Q_s + 0.P &= 0 \end{aligned}\right\} \quad -(2)$$

Now, converting the above given market model into matrix form or notation we have -

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 20 \\ -9 \\ 0 \end{bmatrix}}_{C}$$

or $\quad Ax = C \quad$ ——— (3)

Now applying Cramer's Rule to the above matrix notation, we have:-

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 20 & 0 & 2 \\ -9 & 1 & -2 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{--- (4)}$$

$$Q_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 20 & 2 \\ 0 & -9 & -2 \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{--- (5)}$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & 20 \\ 0 & 1 & -9 \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{--- (6)}$$

$$\therefore |A| = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0-(-2).(-1)) - 0(0.0-(-2).1) + 2(0.(-1)-1.1)$$

$$= 1(0-2) - 0 + 2(0-1) = (-2-2) = -4$$

$$\therefore |A_1| = \begin{vmatrix} 20 & 0 & 2 \\ -9 & 1 & -2 \\ 0 & -1 & 0 \end{vmatrix} = 20\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -9 & -2 \\ 0 & 0 \end{vmatrix} + 2\begin{vmatrix} -9 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= 20(1.0-(-2)(-1)) - 0((-9).0-(-2).0) + 2((-9).(-1)-1.0)$$

$$= 20(0-2) - 0(0+0) + 2(9-0)$$

$$= (-40+18) = -22$$

$$\therefore |A_2| = \begin{vmatrix} 1 & 20 & 2 \\ 0 & -9 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -9 & -2 \\ 0 & 0 \end{vmatrix} - 20\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & -9 \\ 1 & 0 \end{vmatrix}$$

$$= 1((-9).0-(-2).0) - 20(0.0-(-2).1) + 2(0.0-(-9).1)$$

$$= 1(0+0) - 20(0+2) + 2(0+9)$$

$$= (0-40+18) = -22$$

and, $|A_3| = \begin{vmatrix} 1 & 0 & 20 \\ 0 & 1 & -9 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -9 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -9 \\ 1 & 0 \end{vmatrix} + 20\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$

$= 1(1.0-(-9).(-1)) - 0(0.0-(-9).1) + 20(0.(-1)-1.1)$

$= 1(0-9) - 0 + 20(0-1)$

$= (-9-20) = -29$

Now putting the values of $|A|, |A_1|, |A_2|$ and $|A_3|$ in equations ④, ⑤ and ⑥, we have –

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{-22}{-4} = 5{\cdot}5$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{-22}{-4} = 5{\cdot}5$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{-29}{-4} = 7{\cdot}25$$

2. Find out the equilibrium quantity and price of the following market model, if the government introduces a sales tax Rs. 5 per unit of output sold.

$$Q_d = 66 - 3P$$
$$Q_s = -4 + 2P$$
$$Q_d = Q_s$$

Solution: The given market model is

$$\left.\begin{aligned} Q_d &= 66 - 3P \\ Q_s &= -4 + 2P \\ Q_d &= Q_s \end{aligned}\right\} \text{—} (1)$$

It is also given that the government has introduced a sales tax of Rs. 5/- per unit of output sold.

The above given market model can be rewritten as after imposition of the sales tax Rs. 5/- per unit of output

$Q_d = 66 - 3P$

$Q_s = -4 + 2(P-5)$

$Q_d = Q_s$

or, $1.Q_d + 0.Q_s + 3.P = 66$
$0.Q_d + 1.Q_s - 2P = -14$
$1.Q_d - 1.Q_s + 0.P = 0$ ——— (2)

Now converting the above given market model into matrix form or notation, we have –

$$\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 66 \\ -14 \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$

$\Rightarrow X = A^{-1}C$ ——— (3)

We know that $A^{-1} = \frac{Adj'(A)}{|A|}$ and

$Adj(A) = [\text{co-factor matrix of } A]'$

Now,

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0 - (-2).(-1)) - 0(0.0 - (-2).1) + 3(0.(-1) - 1.1)$$

$$= 1(0-2) - 0 + 3(0-1) = (-2-3) = -5$$

$$\text{Co-factor of } A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(1.0-(-2).(-1)) & -(0.0-(-2).1) & +(0.(-1)-1.1) \\ -(0.0-3.(-1)) & +(1.0-3.1) & -(1.(-1)-0.1) \\ +(0.(-2)-3.1) & -(1.(-2)-3.0) & +(1.1-0.0) \end{bmatrix}$$

$$= \begin{bmatrix} +(0-2) & -(0+2) & +(0-1) \\ -(0+3) & +(0-3) & -(-1-0) \\ +(0-3) & -(-2-0) & +(1-0) \end{bmatrix} = \begin{bmatrix} -2 & -2 & -1 \\ -3 & -3 & 1 \\ -3 & 2 & 1 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} -2 & -2 & -1 \\ -3 & -3 & 1 \\ -3 & 2 & 1 \end{bmatrix}' = \begin{bmatrix} -2 & -3 & -3 \\ -2 & -3 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

Now putting the values of $|A|$, $Adj(A)$ and c in equation ③, we have –

$$X = \frac{1}{-5} \begin{bmatrix} -2 & -3 & -3 \\ -2 & -3 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 66 \\ -14 \\ 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} (-2).66+(-3).(-14)+(-3).0 \\ (-2).66+(-3).(-14)+2.0 \\ (-1).66+1.(-14)+1.0 \end{bmatrix}$$

$$= \frac{1}{-5} \begin{bmatrix} (-132+42+0) \\ (-132+42+0) \\ (-66-14+0 \end{bmatrix} = \begin{bmatrix} \frac{-90}{-5} \\ \frac{-90}{-5} \\ \frac{-80}{-5} \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \\ 16 \end{bmatrix}$$

$\therefore \bar{Q}_d = 18$, $\bar{Q}_s = 18$ and $\bar{P} = 16$

Note: If government grants subsidy to the production of a commodity or output, then subsidy will be provided to the producer or supplier directly and it affects directly the supply function of that particular commodity. If government grants subsidy at the rate of 's' per unit of output, then the simple market model for that commodity will be looks like -

$Q_d = a - bP$ (Demand function)

$Q_s = -c + d(P+s)$ (Supply function)

$Q_d = Q_s$ (Equilibrium condition)

Example:

1. Find the equilibrium price and quantity of the following market model, if government grants subsidy Rs. 2/- per unit of output sold.

$Q_d = 100 - P$

$Q_s = -10 + 4P$

$Q_d = Q_s$

Solution: The given market model is

$Q_d = 100 - P$

$Q_s = -10 + 4P$

$Q_d = Q_s$,

It is also given that government grants subsidy Rs. 2/- per unit of output sold.

After introduction of subsidy Rs. 2/- per unit of output sold, the given market model will be looks like -

$$\left.\begin{aligned} Q_d &= 100 - P \\ Q_s &= -10 + 4(P+2) \quad [\because Q_s = -10 + 4(P+s)] \\ Q_d &= Q_s \end{aligned}\right\} \text{--- } (1)$$

The above given market model can be rearitten as -

$1 . Q_d + 0 . Q_s + 1 . P = 100$

$0 . Q_d + 1 . Q_s - 4 . P = -2$

$1 . Q_d - 1 . Q_s + 0 . P = 0$

Now, converting the above market model into matrix form or notation, we have —

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 100 \\ -2 \\ 0 \end{bmatrix}}_{C}$$

or $AX = C$ ——— (2)

Applying Cramer's Rule to the above matrix notation we have —

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 100 & 0 & 1 \\ -2 & 1 & -4 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{——— (3)}$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 100 & 1 \\ 0 & -2 & -4 \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{——— (4)}$$

$$\bar{P} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & 100 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -1 & 0 \end{vmatrix}} \quad \text{——— (5)}$$

$$\therefore |A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -4 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -4 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0-(-4).(-1)) - 0(0.0-(-4).1) + 1(0.(-1)-1.1)$$

$$= 1(0-4) - 0 + 1(0-1) = (-4-1) = -5$$

$$|A_1| = \begin{vmatrix} 100 & 0 & 1 \\ -2 & 1 & -4 \\ 0 & -1 & 0 \end{vmatrix} = 100\begin{vmatrix} 1 & -4 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -2 & -4 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= 100(1.0-(-4).(-1)) - 0((-2).0-(-4).0) + 1((-2).(-1)-1.0)$$

$$= 100(0-4) - 0 + 1(2-0) = (-400+2) = -398$$

$$|A_2| = \begin{vmatrix} 1 & 100 & 1 \\ 0 & -2 & -4 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -2 & -4 \\ 0 & 0 \end{vmatrix} - 100\begin{vmatrix} 0 & -4 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix}$$

$$= 1((-2).0 - (-4).0) - 100(0.0 - (-4).1) + 1(0.0 - (-2).1)$$

$$= 1(0+0) - 100(0+4) + 1(0+2)$$

$$= (0 - 400 + 2) = -398$$

and, $$|A_3| = \begin{vmatrix} 1 & 0 & 100 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} + 100\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.0 - (-2).(-1)) - 0(0.0 - (-2).1) + 100(0.(-1) - 1.1)$$

$$= 1(0+2) - 0 + 100(0-1)$$

$$= (2 - 100) = -98$$

Now, putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$, in equations ③, ④ and ⑤, we have -

$$\bar{Q}_d = \frac{|A_1|}{|A|} = \frac{-398}{-5} = 79{\cdot}6$$

$$\bar{Q}_s = \frac{|A_2|}{|A|} = \frac{-398}{-5} = 79{\cdot}6$$

and $$\bar{p} = \frac{|A_3|}{|A|} = \frac{-98}{-5} = 19{\cdot}6$$