

(B) Cramer's Rule:

In order to derive the Cramer's Rule, let us consider a general form of a linear system of '3' simultaneous equations with '3' endogenous variables x_1, x_2 and x_3 , such that

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \right\} \text{--- (1)}$$

Converting the above equation system into matrix form or notation, we have

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_X = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_C$$

or $AX = C$

$$\Rightarrow X = A^{-1} \cdot C$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where, X is the solution or unknown vector, A is the coefficient matrix of the variables, x_1, x_2 and x_3 , and C is the "right hand" or data vector.

Now, replacing A ^{inverse of} the coefficient matrix by its co-factor matrix, then the above equation system will take the following form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \left[\because A^{-1} = \frac{\text{Adj}^*(A)}{|A|} \text{ and } \text{Adj}^*(A) = (\text{co-factor matrix of } A) \right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} (C_{11} \cdot b_1 + C_{12} \cdot b_2 + C_{13} \cdot b_3) \\ (C_{21} \cdot b_1 + C_{22} \cdot b_2 + C_{23} \cdot b_3) \\ (C_{31} \cdot b_1 + C_{32} \cdot b_2 + C_{33} \cdot b_3) \end{bmatrix} \text{--- (2)}$$

As it is discussed in the previous section (Minors and Co-factors) that

$$C_{11} = (-1)^{1+1} A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (+) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (-) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3} A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (+) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (-) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = (+) \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{23} = (-1)^{2+3} A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = (-) \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{31} = (-1)^{3+1} A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (+) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = (-) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{33} = (-1)^{3+3} A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (+) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

From equation ②, it can be derived that -

$$x_1 = \frac{(b_1 \cdot C_{11} + b_2 \cdot C_{21} + b_3 \cdot C_{31})}{|A|}$$

$$x_2 = \frac{(b_1 \cdot C_{12} + b_2 \cdot C_{22} + b_3 \cdot C_{32})}{|A|}$$

$$x_3 = \frac{(b_1 \cdot C_{13} + b_2 \cdot C_{23} + b_3 \cdot C_{33})}{|A|}$$

} ③

Now putting the values of $C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}$ and C_{33} in equations (3), we have -

$$x_1 = \frac{b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_2 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_3 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}{|A|}$$

$$= \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} \text{ or } \frac{|A_1|}{|A|}$$

Similarly,

$$x_2 = \frac{(-b_1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + b_2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - b_3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}}{|A|}$$

$$= \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|} \text{ or } \frac{|A_2|}{|A|}$$

and,

$$x_3 = \frac{b_1 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - b_2 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + b_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{|A|}$$

$$= \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|} \text{ or } \frac{|A_3|}{|A|}$$

The above results can also be used in a relation $AX = B$ (in matrix form) of n simultaneous equations with n variable such that

$$x_j = \frac{|A_j|}{|A|} = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & b_n & \dots & a_{nn} \end{vmatrix} \quad \left(\because j=1, 2, \dots, n \right)$$

and A is a non-singular matrix)

where the elements $b_1, b_2, \dots, b_n$ are the elements of j -th column of $|A|$ replacing $a_{1j}, a_{2j}, \dots, a_{nj}$.

Example:

1. Solve the following equation system by Cramer's rule.

$$7x_1 - 5x_2 = 11$$

$$3x_1 + 2x_2 = 13$$

Solution: The given equation system is

$$\left. \begin{array}{l} 7x_1 - 5x_2 = 11 \\ 3x_1 + 2x_2 = 13 \end{array} \right\} \text{--- (1)}$$

Now converting the above equation system into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 7 & -5 \\ 3 & -2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 11 \\ 13 \end{bmatrix}}_C$$

$$\text{or } AX = C \quad \text{--- (2)}$$

Applying Cramer's rule to the above matrix notation we have

$$x_1 = \frac{|A_1|}{|A|} \text{ or } \frac{\begin{vmatrix} 11 & -5 \\ 13 & -2 \end{vmatrix}}{\begin{vmatrix} 7 & -5 \\ 3 & -2 \end{vmatrix}}, \text{ and --- (3)}$$

$$x_2 = \frac{|A_2|}{|A|} \text{ or } \frac{\begin{vmatrix} 7 & 11 \\ 3 & 13 \end{vmatrix}}{\begin{vmatrix} 7 & -5 \\ 3 & -2 \end{vmatrix}} \quad \text{--- (4)}$$

$$\text{Now, } |A| = \begin{vmatrix} 7 & -5 \\ 3 & -2 \end{vmatrix} = (7 \cdot (-2) - (-5) \cdot 3) = (-14 + 15) = +1$$

$$|A_1| = \begin{vmatrix} 11 & -5 \\ 13 & -2 \end{vmatrix} = (11 \cdot (-2) - (-5) \cdot 13) = (-22 + 65) = 43$$

$$|A_2| = \begin{vmatrix} 7 & 11 \\ 3 & 13 \end{vmatrix} = (7 \cdot 13 - 11 \cdot 3) = (91 - 33) = 58$$

Now putting the values of $|A|$, $|A_1|$ and $|A_2|$ in eqⁿ (3) and

(4), we have $x_1 = \frac{43}{1} = 43$ and $x_2 = \frac{58}{1} = 58$

2. Solve the following equation system by Cramer's Rule-

$$x_1 + 2x_2 + 3x_3 = 14$$

$$2x_1 + 3x_2 + x_3 = 11$$

$$3x_1 + x_2 + 2x_3 = 11$$

Solution: The given equation system as

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ 2x_1 + 3x_2 + x_3 = 11 \\ 3x_1 + x_2 + 2x_3 = 11 \end{cases} \quad \text{--- (1)}$$

Converting the above equation system into matrix form or notation we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$$

$A \quad X \quad C$

or $AX = C$ --- (2)

Applying Cramer's rule to the above matrix notation we have

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 14 & 2 & 3 \\ 11 & 3 & 1 \\ 11 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}} \quad \text{--- (3)}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 14 & 3 \\ 2 & 11 & 1 \\ 3 & 11 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}} \quad \text{--- (4)}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 14 \\ 2 & 3 & 11 \\ 3 & 1 & 11 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}} \quad \text{--- (5)}$$

Now, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$

$$= 1(3 \cdot 2 - 1 \cdot 1) - 2(2 \cdot 2 - 1 \cdot 3) + 3(2 \cdot 1 - 3 \cdot 3)$$

$$= 1(6 - 1) - 2(4 - 3) + 3(2 - 9) = (1 \times 5) - (2 \times 1) + (3 \times (-7))$$

$$= (5 - 2 - 21) = -18$$

$$\therefore |A_1| = \begin{vmatrix} 14 & 2 & 3 \\ 11 & 3 & 1 \\ 11 & 1 & 2 \end{vmatrix} = 14 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 11 & 1 \\ 11 & 2 \end{vmatrix} + 3 \begin{vmatrix} 11 & 3 \\ 11 & 1 \end{vmatrix}$$

$$= 14(6-1) - 2(22-11) + 3(11-33)$$

$$= (14 \times 5) - (2 \times 11) + (3 \times (-22))$$

$$= (70 - 22 - 66) = -18$$

$$\therefore |A_2| = \begin{vmatrix} 1 & 14 & 3 \\ 2 & 11 & 1 \\ 3 & 11 & 2 \end{vmatrix} = 1 \begin{vmatrix} 11 & 1 \\ 11 & 2 \end{vmatrix} - 14 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 11 \\ 3 & 11 \end{vmatrix}$$

$$= 1(22-11) - 14(4-3) + 3(22-33)$$

$$= (1 \times 11) - (14 \times 1) + (3 \times (-11))$$

$$= (11 - 14 - 33) = -36$$

$$\text{and } |A_3| = \begin{vmatrix} 1 & 2 & 14 \\ 2 & 3 & 11 \\ 3 & 1 & 11 \end{vmatrix} = 1 \begin{vmatrix} 3 & 11 \\ 1 & 11 \end{vmatrix} - 2 \begin{vmatrix} 2 & 11 \\ 3 & 11 \end{vmatrix} + 14 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1(33-11) - 2(22-33) + 14(2-9)$$

$$= (1 \times 22) - (2 \times (-11)) + (14 \times (-7))$$

$$= (22 + 22 - 98) = -54$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations ③, ④ and ⑤, we have -

$$x_1 = \frac{|A_1|}{|A|} = \frac{-18}{-18} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-36}{-18} = 2$$

$$\text{and } x_3 = \frac{|A_3|}{|A|} = \frac{-54}{-18} = 3$$

Q. Solve the equation system by Cramer's rule.

$$p + 4q + 2r = 36$$

$$4p + 5q + r = 35$$

$$2p + 3q + 2r = 35$$

$$\therefore \begin{vmatrix} 1 & 4 & 2 \\ 4 & 5 & 1 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 2 \\ 4 & 5 & 1 \\ 2 & 3 & 2 \end{vmatrix} = |A|$$

$$((1 \times 5 \times 2) + (4 \times 1 \times 2) + (2 \times 4 \times 2)) - ((1 \times 1 \times 2) + (4 \times 2 \times 2) + (2 \times 4 \times 1))$$

Solution: The given equation system is

$$\left. \begin{aligned} p+4q+2r &= 36 \\ 4p+5q-r &= 35 \\ 2p+3q+2r &= 35 \end{aligned} \right\} \text{--- (1)}$$

converting the above given equation system into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & 4 & 2 \\ 4 & 5 & -1 \\ 2 & 3 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} p \\ q \\ r \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 36 \\ 35 \\ 35 \end{bmatrix}}_C$$

$$\text{or } AX = C \text{ --- (2)}$$

Now applying Cramer's Rule to the above matrix notation we have -

$$p = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 36 & 4 & 2 \\ 35 & 5 & -1 \\ 35 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 2 \\ 4 & 5 & -1 \\ 2 & 3 & 2 \end{vmatrix}} \text{ --- (3)}$$

$$q = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 36 & 2 \\ 4 & 35 & -1 \\ 2 & 35 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 2 \\ 4 & 5 & -1 \\ 2 & 3 & 2 \end{vmatrix}} \text{ --- (4)}$$

$$r = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 4 & 36 \\ 4 & 5 & 35 \\ 2 & 3 & 35 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 2 \\ 4 & 5 & -1 \\ 2 & 3 & 2 \end{vmatrix}} \text{ --- (5)}$$

$$\begin{aligned} \therefore |A| &= \begin{vmatrix} 1 & 4 & 2 \\ 4 & 5 & -1 \\ 2 & 3 & 2 \end{vmatrix} = 1 \begin{vmatrix} 5 & -1 \\ 3 & 2 \end{vmatrix} - 4 \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} \\ &= 1(10+3) - 4(8+2) + 2(12-10) \\ &= (1 \times 13) - (4 \times 10) + (2 \times 2) = (13 - 40 + 4) = -23 \end{aligned}$$

$$\begin{aligned} \therefore |A_1| &= \begin{vmatrix} 36 & 4 & 2 \\ 35 & 5 & -1 \\ 35 & 3 & 2 \end{vmatrix} = 36 \begin{vmatrix} 5 & -1 \\ 3 & 2 \end{vmatrix} - 4 \begin{vmatrix} 35 & -1 \\ 35 & 2 \end{vmatrix} + 2 \begin{vmatrix} 35 & 5 \\ 35 & 3 \end{vmatrix} \\ &= 36(10+3) - 4(70+35) + 2(105-175) \\ &= (36 \times 13) - (4 \times 105) + (2 \times (-70)) \\ &= (468 - 420 - 140) = -92 \end{aligned}$$

$$\therefore |A_2| = \begin{vmatrix} 1 & 36 & 2 \\ 4 & 35 & -1 \\ 2 & 35 & 2 \end{vmatrix} = 1 \begin{vmatrix} 35 & -1 \\ 35 & 2 \end{vmatrix} - 36 \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 35 \\ 2 & 35 \end{vmatrix}$$

$$= 1(70 + 35) - 36(8 + 2) + 2(140 - 70)$$

$$= (1 \times 105) - (36 \times 10) + (2 \times 70)$$

$$= (105 - 360 + 140) = -115$$

$$\text{and } |A_3| = \begin{vmatrix} 1 & 4 & 36 \\ 4 & 5 & 35 \\ 2 & 3 & 35 \end{vmatrix} = 1 \begin{vmatrix} 5 & 35 \\ 3 & 35 \end{vmatrix} - 4 \begin{vmatrix} 4 & 35 \\ 2 & 35 \end{vmatrix} + 36 \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$$

$$= 1(175 - 105) - 4(140 - 70) + 36(12 - 10)$$

$$= (1 \times 70) - (4 \times 70) + (36 \times 2)$$

$$= (70 - 280 + 72) = -138$$

Now putting the value at $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations (3), (4) and (5), we have

$$p = \frac{|A_1|}{|A|} = \frac{-92}{-23} = 4$$

$$q = \frac{|A_2|}{|A|} = \frac{-115}{-23} = 5$$

$$r = \frac{|A_3|}{|A|} = \frac{-138}{-23} = 6$$

4. Solve the following equation system by Cramer's rule.

$$x_1 + 2x_2 - x_3 = 6$$

$$2x_1 - x_2 - 2x_3 = 3$$

$$4x_1 + 3x_2 + x_3 = 9$$

Solution: The given equation system is

$$\left. \begin{aligned} x_1 + 2x_2 - x_3 &= 6 \\ 2x_1 - x_2 - 2x_3 &= 3 \\ 4x_1 + 3x_2 + x_3 &= 9 \end{aligned} \right\} \text{--- (1)}$$

Converting the ^{above} equation system into matrix form or notation, we have -

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}}_C$$

or $AX = C$ ————— (2)

Now applying Cramer's Rule to the above matrix notation, we have -

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 6 & -2 & -1 \\ 3 & -1 & -2 \\ 9 & -3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{vmatrix}} \quad \text{--- (3)}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 6 & -1 \\ 3 & 3 & -2 \\ 4 & 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{vmatrix}} \quad \text{--- (4)}$$

and,

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & -6 \\ 3 & -1 & 3 \\ 4 & 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{vmatrix}} \quad \text{--- (5)}$$

$$\begin{aligned} \therefore |A| &= \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{vmatrix} = 1 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} \\ &= 1(-1+6) - 2(3+8) - 1(9+4) \\ &= (1 \times 5) - (2 \times 11) - (1 \times 13) \\ &= (5 - 22 - 13) = -30 \end{aligned}$$

$$\begin{aligned} \therefore |A_1| &= \begin{vmatrix} 6 & 2 & -1 \\ 3 & -1 & -2 \\ 9 & 3 & 1 \end{vmatrix} = 6 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ 9 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 3 & -1 \\ 9 & 3 \end{vmatrix} \\ &= 6(-1+6) - 2(3+18) - 1(9+9) \\ &= (6 \times 5) - (2 \times 21) - (1 \times 18) = (30 - 42 - 18) = -30 \end{aligned}$$

$$\therefore |A_2| = \begin{vmatrix} 1 & 6 & -1 \\ 3 & 3 & -2 \\ 4 & 9 & 1 \end{vmatrix} = 1 \begin{vmatrix} 3 & -2 \\ 9 & 1 \end{vmatrix} - 6 \begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 3 \\ 4 & 9 \end{vmatrix}$$

$$= 1(3+18) - 6(3+8) - 1(27-12)$$

$$= (1 \times 21) - (6 \times 11) - (1 \times 15)$$

$$= (21 - 66 - 15) = -60$$

$$\text{and } |A_3| = \begin{vmatrix} 1 & 2 & 6 \\ 3 & -1 & 3 \\ 4 & 3 & 9 \end{vmatrix} = 1 \begin{vmatrix} -1 & 3 \\ 3 & 9 \end{vmatrix} - 2 \begin{vmatrix} 3 & 3 \\ 4 & 9 \end{vmatrix} + 6 \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 1(-9-9) - 2(27-12) + 6(9+4)$$

$$= (1 \times (-18)) - (2 \times 15) + (6 \times 13)$$

$$= (-18 - 30 + 78) = 40$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equations (3), (4) and (5), we have

$$x_1 = \frac{|A_1|}{|A|} = \frac{-30}{-50} = \frac{3}{5}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-60}{-50} = \frac{6}{5}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{40}{-50} = -\frac{4}{5}$$

5. Solve the following equation system by Cramer's rule -

$$\begin{cases} 2x_1 + 3x_2 + 3x_3 = 5 \\ 3x_1 - x_2 - 2x_3 = 3 \\ x_1 - 2x_2 + x_3 = -4 \end{cases}$$

Solution: The given equation system is

$$\begin{cases} 2x_1 + 3x_2 + 3x_3 = 5 \\ 3x_1 - x_2 - 2x_3 = 3 \\ x_1 - 2x_2 + x_3 = -4 \end{cases} \quad \text{--- (1)}$$

Converting the above give equation system into matrix form or notation, we have -

$$\begin{bmatrix} 2 & 3 & 3 \\ 3 & -1 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}$$

A X C

or

$$AX = C \quad \text{--- (2)}$$

Now applying Cramer's Rule to the above matrix notation, we have -

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 5 & 3 & 3 \\ 3 & -1 & -2 \\ -4 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 3 \\ 3 & -1 & -2 \\ 1 & -2 & 1 \end{vmatrix}} \quad \text{--- (3)}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 3 & 3 & -2 \\ 1 & -4 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 3 \\ 3 & -1 & -2 \\ 1 & -2 & 1 \end{vmatrix}} \quad \text{--- (4)}$$

$$\text{and, } x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 2 & 3 & 5 \\ 3 & -1 & 3 \\ 1 & -2 & -4 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 3 \\ 3 & -1 & -2 \\ 1 & -2 & 1 \end{vmatrix}} \quad \text{--- (5)}$$

$$\begin{aligned} \therefore |A| &= \begin{vmatrix} 2 & 3 & 3 \\ 3 & -1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = 2 \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} \\ &= 2(-1-4) - 3(3+2) + 3(-6+1) \\ &= (2 \times (-5)) - (3 \times 5) + (3 \times (-5)) \\ &= (-10 - 15 - 15) = -40 \end{aligned}$$

$$\begin{aligned} \therefore |A_1| &= \begin{vmatrix} 5 & 3 & 3 \\ 3 & -1 & -2 \\ -4 & -2 & 1 \end{vmatrix} = 5 \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -1 \\ -4 & -2 \end{vmatrix} \\ &= 5(-1-4) - 3(3-8) + 3(-6-4) \\ &= (5 \times (-5)) - (3 \times (-5)) + (3 \times (-10)) \\ &= (-25 + 15 - 30) = -40 \end{aligned}$$

$$\therefore |A_2| = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 3 & -2 \\ 1 & -4 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} - 5 \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 3 \\ 1 & -4 \end{vmatrix}$$

$$= 2(3-8) - 5(3+2) + 3(-12-3)$$

$$= (2 \times (-5)) - (5 \times 5) + (3 \times (-15))$$

$$= (-10 - 25 - 45) = -80$$

$$\text{and } |A_3| = \begin{vmatrix} 2 & 3 & 5 \\ 3 & -1 & 3 \\ 1 & -2 & -4 \end{vmatrix} = 2 \begin{vmatrix} -1 & 3 \\ -2 & -4 \end{vmatrix} - 3 \begin{vmatrix} 3 & 3 \\ 1 & -4 \end{vmatrix} + 5 \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 2(4+6) - 3(-12-3) + 5(-6+2)$$

$$= (2 \times 10) - (3 \times (-15)) + (5 \times (-5))$$

$$= (20 + 45 - 25) = 40$$

Now putting the values of $|A|$, $|A_1|$, $|A_2|$ and $|A_3|$ in equation ③, ④ and ⑤, we have

$$x_1 = \frac{|A_1|}{|A|} = \frac{-40}{-40} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-80}{-40} = 2$$

$$\text{and } x_3 = \frac{|A_3|}{|A|} = \frac{40}{-40} = -1$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} y + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A| \therefore$$

$$(1+0+0)x + (0+0+0)y + (0+0+0)z =$$

$$((1) \times 1) + ((0) \times 1) + ((0) \times 1) =$$

$$1+0+0 = (1+0+0) =$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} y + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A| \therefore$$

$$(1+0+0)x + (0+0+0)y + (0+0+0)z =$$

$$((1) \times 1) + ((0) \times 1) + ((0) \times 1) =$$