## Important concepts of Matrix:

**Rank of a Matrix:** The order of the largest non-singular square submatrix (which is contained in a matrix), is termed as Rank of a Matrix. If $A$ is a matrix, then rank of $A$ is denoted by $\rho(A)$.

The following procedure is generally adopted for finding rank of a matrix:

(a) Calculate determinant value of the largest square submatrix or the minors of highest possible order. If the determinant value is not zero, then order of the submatrix or minor is the rank of the matrix.

(b) If the determinant value is zero, and every other minors of same order is zero, then calculate classmate

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minor ~~of~~ or square sub-matrix of next ~~e~~ lower order. If the determinant value of any one minor of this lower order is not equal to zero, then this order will be the rank of the original matrix.

(c) If the determinant values of all the minors of this lower order are zero, then calculate determinant value of next ~~order~~ lower order minors and so on.

*** Example:

(1) If $A = \begin{bmatrix} 6 & 5 \\ 1 & 3 \end{bmatrix}_{(2\times2)}$, find the rank of A.

Solution: It is given that $A = \begin{bmatrix} 6 & 5 \\ 1 & 3 \end{bmatrix}_{(2\times2)}$

$$\therefore |A| = \begin{vmatrix} 6 & 5 \\ 1 & 3 \end{vmatrix} = (6.3 - 5.1) = (18-5) = 13$$

$\therefore$ Rank of A $(\rho(A)) = 2$, because the highest order of the square matrix is $(2\times2)$ and determinant value of the highest order square matrix ~~(|A|)~~ is not equal to zero, i.e., $|A| \neq 0$.

(2) If $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, find the rank of A.

Solution: It is given that

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}_{(2\times2)}$$

$$\therefore |A| = (2.3 - 1.6) = (6-6) = 0$$

Since, $|A| = 0$, hence A is a singular matrix and rank of A will not be 2. The order of the original matrix is $(2\times2)$. Now ~~it is considered~~ the next lower order square submatrix or minor is considered for find out the rank of A.

$$\therefore A_1 = [2]_{(1\times1)}, \therefore |A_1| = 2 \neq 0$$

Hence, rank of $A\ [\rho(A)] = 1$, because the determinant value of the lower order square submatrix is not equal to zero $(|A_1| \neq 0)$ and order of this submatrix is $(1\times1)$.

(3) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$, then find rank of A.

Solution: It is given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}_{(3\times3)}$$

$$\therefore |A| = 1\begin{vmatrix} 4 & 7 \\ 6 & 10 \end{vmatrix} - 2\begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$$

$$= 1(4.10 - 7.6) - 2(2.10 - 7.3) + 3(2.6 - 4.3)$$

$$= 1(40 - 42) - 2(20 - 21) + 3(12 - 12)$$

$$= (1\times(-2)) - (2\times(-1)) + (3\times0)$$

$$= (-2 + 2 + 0) = 0$$

Since, $|A| = 0$, hence A is a singular matrix and rank of A will not be 3. The order of the original matrix A is $(3\times3)$.

Now, the next lower order square submatrix or minor is considered for find out the rank of A.

$$\therefore A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{(2\times2)}, \quad |A_1| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (4-4) = 0$$

$$A_2 = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}_{(2\times2)}, \quad |A_2| = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = (14-12) = 2 \neq 0$$

Hence rank of A $[\rho(A)] = 2$, because the determinant value of this lower order square sub-matrix is not equal to zero $(|A_2| \neq 0)$ and order of this lower submatrix is $(1\times1)$.

*** If A is a matrix of order $(n\times n)$ and rank of A $[\rho(A)]$ is less than 'n', then the inverse of A does not exist because the determinant value of the matrix will be zero. This type of matrix is known as 'Singular Matrix'.

**Norm of a Matrix:** The norm of a matrix is defined as the largest column sum of the matrix.
For example:

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & 10 \\ 3 & 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 2 & 0 & 3 \\ 3 & 0 & 2 & 2 \\ 0 & 5 & 7 & 9 \\ 5 & 3 & 2 & 7 \end{bmatrix}$$

The largest column sum of matrix A is 18 relating to third column and in matrix B, the largest column sum is 21 relating to fourth column. So the norm of A is 18 and the norm of B is 21.

**Theorem related to norm of a Matrix:** If any two matrices (say A and B) are conformable for multiplication, then norm of the product of the matrices (N(AB)) is less than or equal to the product of individual norm of the matrices (i.e., N(A)×N(B)). Symbolically,

$$N(AB) \leqslant N(A) \times N(B)$$

**Proof:** if $A = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}_{(2\times2)}$ and $B = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}_{(2\times2)}$

then, Norm of $A\,(N(A)) = 6$ relating to 1st column

Norm of $B\,(N(B)) = 7$ relating to 1st column

Therefore, $N(A) \times N(B) = (6 \times 7) = 42$ —— ①

Now $AB = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (5.2+3.5) & (5.1+3.2) \\ (1.2+2.5) & (1.1+2.2) \end{bmatrix} = \begin{bmatrix} (10+15) & (5+6) \\ (2+7) & (1+4) \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 11 \\ 9 & 5 \end{bmatrix}$$

∴ Norm of $AB\,(N(AB)) = 34$ relating to 1st column ②

By comparing equation ① and ②, it can be said that $N(AB) \leqslant N(A) \times N(B)$ [∵ $34 < 42$]

Hence proved it.

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**Trace of a Matrix:** The trace of a matrix is the sum of its diagonal elements. If $A$ is a square matrix of order $(n \times n)$, then trace of $A$ ~~can~~ $[tr(A)]$ can be defined as -

$$tr[A] = \sum_{i=1}^{n} a_{ii}$$

For example, if $A = \begin{bmatrix} 15 & 6 & 7 \\ 3 & 9 & 2 \\ -1 & 3 & -7 \end{bmatrix}_{(3 \times 3)}$

$$\therefore tr[A] = \sum_{i=1}^{3} a_{ii}$$

$$= a_{11} + a_{22} + a_{33}$$

$$= 15 + 9 + (-7)$$

$$= 17$$

**Properties of Trace of Matrix:**

(a) $tr(A+B) = tr(A) + tr(B)$

(b) $tr(A') = tr(A)$

(c) $tr(AB) = tr(BA)$