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Properties of OLS estimators (multiple regression):

(1) Property of linearity: The OLS estimators $\hat{\beta}$ is given by

$$\hat{\beta} = (X'X)^{-1} X'Y$$

Let, $A = (X'X)^{-1} X'$ is a matrix of order $K \times n$

$$\hat{B} = AY$$

$\hat{\beta}$ is a linear function of Y since A is a $K \times n$ matrix of fix elements.

(2) Property of unbaisedness:- We have the OLS estimator —

$$\hat{\beta} = (X'X)^{-1} X'Y \quad \text{— (1)}$$

We know that $Y = X\beta + U \longrightarrow$ (2)

Now, substituting equation (2) in (1)

$$\hat{\beta} = (X'X)^{-1} X'(X\beta + U)$$

$$= (X'X)^{-1} X'XB + (X'X)^{-1} X'U$$

$$= \beta + (X'X)^{-1} X'U$$

Now, $E(\hat{B}) = \beta + (X'X)^{-1} X' E(U)$

$$= \beta \qquad \left[\because E(U) = 0\right.$$

Thus, the OLS estimator $\hat{\beta}$ is an unbaised estimator of β.

(3) Property of minimum variance:- Here, we have to prove that the OLS estimator $\hat{\beta}$ has the smallest variance. For this we have to first obtain the ~~variance~~ var-cov$(\hat{\beta})$.

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$$\text{var} - \text{cov}(\hat{\beta}) = E\left[\{\hat{\beta} - E(\hat{\beta})\}\{\hat{\beta} - E(\hat{\beta})\}'\right]$$

$$= E\left[\{\hat{\beta} - \beta\}\{\hat{\beta} - \beta\}'\right]$$

$$= E\left[\{(X'X)^{-1}X'U\}\{(X'X)^{-1}X'U\}'\right]$$

$$= E\left[(X'X)^{-1}X'UU'X(X'X)^{-1}\right]$$

$$= \left[(X'X)^{-1}X'E(UU')X(X'X)^{-1}\right]$$

$$= (X'X)^{-1}X'\sigma^2 I_n X(X'X)^{-1}$$

$$= \sigma^2(X'X)^{-1}X' I_n X(X'X)^{-1} \quad [\because I_n X = X]$$

$$= \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$

$$= \sigma^2(X'X)^{-1}$$

[In order to compare, let us consider b is an arbitary estimates of β such that

$$b = (C + A)Y \longrightarrow (1)$$

$A = (X'X)^{-1}X'$ and C is a matrix of order $(k \times n)$]

Since b is a linear function of Y therefore, substituting $Y = X\beta + U$ in (1) we get]

$$b = (C + A)(X\beta + U)$$

$$= CX\beta + CU + AX\beta + AU$$

$$= CX\beta + AX\beta + CU + AU$$

$$= (CX + AX)\beta + (C + A)U \text{ —— } (2)$$

Now, $E(b) = E\left[(CX + AX)\beta + (C + A)U\right]$

$$= \left[(CX + AX)\beta + (C + A)E(U)\right]$$

$$= (CX + AX)\beta \qquad \because AX = (X'X)^{-1}X'X = I$$

$$= (CX + I)\beta$$

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$\therefore E(b) = \beta$ iff $CX = 0$ [iff → if and only if

Thus b is an unbaised estimator of β.]

From equation ② if $CX = 0$ then we get

$$= (AX)\beta + (C + A)U$$
$$= \beta + (C+A)U$$

Now,

$$\text{var-cov}(b) = E\left[\{b - E(b)\}\{b - E(b)\}'\right]$$
$$= E\left[\{b-\beta)(b-\beta)'\right]$$
$$= E\left[\{(C+A)U\}\{(C+A)U\}'\right]$$

$\because b - \beta = (C+A)U$

$$= E\left[(C+A)UU'(C+A)'\right]$$
$$= \left[(C+A)E(UU')(C+A)'\right]$$
$$= \{(C+A)\sigma^2 I_n (C+A)'\}$$
$$= \sigma^2 (C+A)(C'+A') \quad [\because I_n = 1$$
$$= \sigma^2 (CC' + CA' + AC' + AA')$$
$$= \sigma^2 . (CC' + AA')$$
$$= \sigma^2 CC' + \sigma^2 . AA'$$
$$= \sigma^2 CC' + \sigma^2 (X'X)^{-1}$$
$$= \sigma^2 . CC' + \text{var-cov}(\hat{\beta})$$

$\because AA' = (X'X)^{-1} . X' \{(X'X)^{-1} . X'\}'$
$= (X'X)^{-1} . X'(X'X)^{-1} . X$
$= (X'X)^{-1} . X'X . (X'X)^{-1}$
$= (X'X)^{-1}$

$AA' = \cancel{(X'X)^{-1}X'} \cancel{\{(X'X)^{-1}X\}}$

Thus, $\text{var-cov}(b) = \sigma^2 . CC' + \text{var-cov}(\hat{\beta})$; which means $\text{var-cov}(\hat{\beta}) < \text{var-cov}(b)$. Thus OLS estimator passess the smallest variance. Hence it is the best estimator. This is also known as the Gauss-Markov theoream.